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TYPOTEX

BÁLINT TÓTH

LIMIT THEOREMS OF PROBABILITY THEORY

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Abstract
Contents
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This course is taught regularly for those MSc students of mathematics at TU Budapest who chose stochastics as topics of specialization. It is assumed that students have a solid background in probability theory (with measure theoretic foundations) and analysis.

The following material covers: ergodic theorems (von Neumann's and Birkhoff's); limit theorems “with bare hands”: Levy's arcsine laws, sojourn time and local time of 1d random walk; the method of moments with applications; the method of characteristic functions: Lindeberg's theorem with applications, Erdős–Kac theorem (CLT for the number of prime divisors), various other applications; stable laws and stable limits with applications; infinitely divisible distributions, Lévy–Khinchin formula and elements of Lévy processes. With lots of problems for solution and applications.

Key words and phrases: Ergodic theorems, limit theorems, characteristic functions, Lindeberg's theorem, Erdős–Kac theorem, stable laws, infinitely divisible distributions, Lévy–Khinchin formula.

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Chapter 1

Stationary sequences, ergodic theorems

1.1 Stationary sequences of random variables

- $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space
- $(S, \mathcal{S})$ a measurable space
- $\xi_j : \Omega \to S$ measurable functions, $j \in \mathbb{N}$ (or $j \in \mathbb{Z}$)

Definition 1.1. *The sequence of (S-valued) random variables ξ_j is stationary iff $(\forall k \in \mathbb{N})$ (or $(\forall k \in \mathbb{Z})$) and $(\forall l \geq 0)$:*

$$distrib\,(\xi_0, \xi_1, \ldots, \xi_l) = distrib\,(\xi_k, \xi_{k+1}, \ldots, \xi_{k+l})$$

Elementary remarks:

Remark 1.1. *A stationary sequence $(\xi_j)_{j\in\mathbb{N}}$ can always be embedded into a stationary sequence $(\xi_j)_{j\in\mathbb{Z}}$.*

Remark 1.2. *If $(\xi_j)_{j\in\mathbb{Z}}$ is a stationary sequence of $(S,\mathcal{S})$-valued random variables, $(\widetilde{S},\widetilde{\mathcal{S}})$ is another measurable space, $g : S^{\mathbb{Z}} \to \widetilde{S}$ is measurable map, and*

$$\widetilde{\xi}_j := g(\dots,\xi_{j-1},\xi_j,\xi_{j+1},\dots).$$

Then: $\left(\widetilde{\xi}_j\right)_{j\in\mathbb{Z}}$ is a stationary sequence of $(\widetilde{S},\widetilde{\mathcal{S}})$-valued random variables.

The essential content of ergodic theorems: generalizations of the laws of large numbers.

If $(X_j)_{j=0}^{\infty}$ is a stationary sequence of $\mathbb{R}$-valued random variables, such that $\mathbf{E}(|X_j|) < \infty$, then

$$\frac{1}{n}\sum_{j=0}^{n-1} X_j \to \mathbf{E}(X_1)$$

asymptotic time averages = state-space averages

– almost surely and in $L^1(\Omega,\mathcal{F},\mathbf{P})$ (Birkhoff, difficult);

– in $L^2(\Omega,\mathcal{F},\mathbf{P})$, (von Neumann, easier).

1.1.1 Examples of stationary sequences

Example 1.1 (I.i.d. sequences). $(\xi_j)_{j\in\mathbb{Z}}$ *i.i.d. sequence of $(S,\mathcal{S})$-valued random variables.*

Example 1.2 (Finitely dependent sequences). *Let $(\xi_j)_{j\in\mathbb{Z}}$ i.i.d. sequence of $(S,\mathcal{S})$-valued random variables, $(\widetilde{S},\widetilde{\mathcal{S}})$ another measurable space, $m \geq 0$ (fixed), $g : S^{m+1} \to \widetilde{S}$ measurable map. Then*

$$\widetilde{\xi}_j := g(\xi_j,\dots,\xi_{j+m})$$

is a $(\widetilde{S},\widetilde{\mathcal{S}})$-valued stationary sequence.

E.g. ξ_j i.i.d. Bernoulli, $\widetilde{\xi}_j := \max\{\xi_j,\xi_{j+1}\}$.

Example 1.3 (Example 3a, 3b). $(\xi_j)_{j\in\mathbb{Z}}$ *i.i.d. Bernoulli,* $\mathbf{P}(\xi_j = 0) = 1/2 = \mathbf{P}(\xi_j = 1)$.

$$\zeta_j := \sum_{k=0}^{\infty} 2^{-k-1}\xi_{j+k}$$
$$\eta_j := \sum_{k=0}^{\infty} 2^{-k-1}\xi_{j-k}$$

Then:

$$\mathit{distrib}(\zeta_j) = UNI[0,1] = \mathit{distrib}(\eta_j).$$

Remarks:

$$\zeta_{j+1} = \{2\zeta_j\} := 2\zeta_j - [2\zeta_j] \quad \textit{deterministically!}$$
$$(\eta_j)_{j\geq 0} \quad \textit{is a Markov chain on} \quad [0,1].$$

Example 1.4 (Stationary Markov chains). *Let* S *be a finite or countable state space,* $P = (P_{\alpha,\beta})_{\alpha,\beta\in S}$ *stochastic matrix,* $\pi : S \to [0,1]$, $\sum_{\alpha\in S}\pi(\alpha) = 1$ *stationary for* P*:*

$$\sum_{\alpha\in S}\pi(\alpha)P_{\alpha,\beta} = \pi(\beta).$$

$(\xi_j)_{j\geq 0}$ *the stationary Markov chain:*

$$\mathbf{P}(\xi_0 = \alpha_0, \xi_1 = \alpha_1, \dots, \xi_l = \alpha_l) = \pi(\alpha_0)P_{\alpha_0,\alpha_1}\dots P_{\alpha_{l-1},\alpha_l}$$

Example 1.5 (Rotations of the circle). $S = [0,1)$, $\mathcal{S} =$ *Borel,* $\mathbf{P} =$ *Lebesgue.*

$$\theta \in (0,1) \text{ (fixed)}, \qquad \xi_j(\omega) := \{\omega + \theta\}, \quad j \in \mathbb{Z}$$

Example 1.6 ("Bernoulli shift"). *(see also Example 3a)* $S = [0,1)$, $\mathcal{S} =$ *Borel,* $\mathbf{P} =$ *Lebesgue.*

$$\xi_j(\omega) := \{2^j\omega\} = 2^j\omega - [2^j\omega], \qquad j \geq 0$$

1.1.2 Measure preserving transformations, dynamical systems

Definition 1.2. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. The $T : \Omega \to \Omega$ measurable transformation is measure preserving if*

$$\forall A \in \mathcal{F} : \qquad \mathbf{P}\left(T^{-1}A\right) = \mathbf{P}(A).$$

We call $(\Omega, \mathcal{F}, \mathbf{P}, T)$ an endomorphism or a dynamical system.
If T is a.s. invertible we call it an automorphism.

Let $(S, \mathcal{S})$ be another measurable space and $g : \Omega \to S$ a measurable function. Then

$$\xi_j := g(T^j \omega)$$

is a stationary sequence of S-valued random variables.

Remark 1.3. *Any stationary sequence of random variables can be realized this way!*

$(S, \mathcal{S})$ measurable space, $(\xi_j)_{j=0}^{\infty}$ stationary sequence of S-valued random variables.

$$\begin{aligned}
&\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\} \\
&\mathcal{F} := \sigma\big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\big) \\
&\mathbf{P} = \text{joint distribution of } (\xi_j)_{j=0}^{\infty} \\
&T : \Omega \to \Omega, \qquad \big(T\omega\big)_j = \omega_{j+1} \\
&g : \Omega \to S, \qquad g(\omega) := \omega_0
\end{aligned}$$

1.1.3 The invariant sigma-algebra, ergodicity

Definition 1.3. *Let $(\Omega, \mathcal{F}, \mathbf{P}, T)$ be an endomorphism. Then*

$$\mathcal{I} := \{A \in \mathcal{F} : \mathbf{P}\left(A \circ T^{-1}A\right) = 0\} \subset \mathcal{F}$$

is the sub-sigma-algebra of invariant sets.

Definition 1.4. *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic iff the invariant sigma-algebra* $\mathcal{I}$ *is trivial with respect to* $\mathbf{P}$*:*

$$\forall A \in \mathcal{I} : \qquad \mathbf{P}(A) \in \{0, 1\}.$$

Remark 1.4. Equivalently: $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic iff for* $f : \Omega \to \mathbb{R}$ *measurable*

$$\{f(T\omega) = f(\omega) \ a.s.\} \Leftrightarrow \{f(\omega) = const. \ a.s.\}$$

Example 1.7 (I.i.d. sequence). *See Example 1.1.* $(S, \mathcal{S}, \mathbf{P}_1)$ *a probability space,*

$$\begin{aligned}
&\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\} \\
&\mathcal{F} := \sigma\big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\big) \\
&\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1 \times \dots \\
&T : \Omega \to \Omega, \qquad \big(T\omega\big)_j = \omega_{j+1}
\end{aligned}$$

Theorem 1.1. *The endomorphism* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

Proof. The tail sigma-algebra is

$$\mathcal{T} := \bigcap_n \sigma(\omega_n, \omega_{n+1}, \omega_{n+2}, \dots)$$

Fact: $\mathcal{I} \subset \mathcal{T}$. Not very difficult.

Kolmogorov's 0-1 law: $\mathcal{T}$ is $\mathbf{P}$-trivial. □

Example 1.8 (Factors). *See Example 1.2 and Example 1.3* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *and* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$ *dynamical systems,* $\varphi : \Omega \to \widetilde{\Omega}$ *measurable, such that*

$$\begin{aligned}
&\mathbf{P}(\varphi^{-1}(A)) = \widetilde{\mathbf{P}}(A) &&\forall A \in \widetilde{\mathcal{F}} \\
&\varphi \circ T = \widetilde{T} \circ \varphi &&\mathbf{P} - a.s.
\end{aligned}$$

then $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$ *is a factor of* $(\Omega, \mathcal{F}, \mathbf{P}, T)$.

Theorem 1.2. *If $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$ is a factor of $(\Omega, \mathcal{F}, \mathbf{P}, T)$ and $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic then so is $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$.*

Proof. Homework. □

Example 1.9 (Ergodic Markov chains). *See Example 1.4. The state space: $(S, \mathcal{S})$ finite or countable*

The stochastic matrix $P = (P_{\alpha,\beta})_{\alpha,\beta \in S}$,

π probability measure on S, stationary for P: $\pi P = \pi$.

$$\begin{aligned}
&\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\} \\
&\mathcal{F} := \sigma\big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\big) \\
&\mathbf{P}\left(\omega_0, \omega_1, \dots, \omega_l\right) = \pi(\omega_0) P_{\omega_0,\omega_1} \dots P_{\omega_{l-1},\omega_l} \\
&T : \Omega \to \Omega, \qquad \big(T\omega\big)_j = \omega_{j+1}
\end{aligned}$$

Theorem 1.3. *The dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic iff P is irreducible.*

Proof. : Proof of $\Rightarrow$: trivial

Proof of $\Leftarrow$: Denote $\mathcal{F}_n := \sigma(\omega_0, \dots, \omega_n)$ and let $A \in \mathcal{I}$.

Then $\mathbf{E}\left(1\!\!1_A \,\middle|\, \mathcal{F}_n\right)$ is a bdd martingale w.r.t. the filtration $\mathcal{F}_n$ and

$$\mathbf{E}\left(1\!\!1_A \,\middle|\, \mathcal{F}_n\right)(\omega) \overset{(1)}{=} \mathbf{E}\left(1\!\!1_A \circ T^n \,\middle|\, \mathcal{F}_n\right)(\omega) \overset{(2)}{=} h(\omega_n)$$

(1): due to invariance of A

(2): due to the Markov property

Due to the martingale convergence theorem

$$h(\omega_n) = \mathbf{E}\left(1\!\!1_A \,\middle|\, \mathcal{F}_n\right)(\omega) \xrightarrow{\text{a.s.}} \mathbf{E}\left(1\!\!1_A \,\middle|\, \mathcal{F}_\infty\right)(\omega) = 1\!\!1_A(\omega)$$

This can hold only if $h \equiv \text{const.}$ □

Example 1.10 (Rotations of the circle). *See Example 1.5. $\Omega = [0, 1)$, $\mathcal{F} =$ Borel, $\mathbf{P} =$ Lebesgue, $T\omega := \{\omega + \theta\}$*

Theorem 1.4. *The dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic iff θ is irrational.*

Proof. Fourier method: let $f \in L^2(\Omega, \mathcal{F}, \mathbf{P})$.

$$f(\omega) \stackrel{L^2}{=} \sum_{k=-\infty}^{\infty} c_k e^{i2\pi k\omega}, \qquad c_k = \int_0^1 e^{-i2\pi k\omega} f(\omega) d\omega$$

Then

$$\begin{aligned} \{f(\omega) = f(T\omega) \quad \text{a.s.}\} &\Leftrightarrow \{\forall k \in \mathbb{Z}: \ c_k \left(e^{i2\pi k\theta} - 1\right) = 0\} \\ &\Leftrightarrow \left\{ \begin{array}{ll} \theta \notin \mathbb{Q}: & c_k = \delta_{k,0} \\ \theta = \frac{p}{q} \in \mathbb{Q}: & c_k = c_k 1\!\!1_{\{k=mq\}} \end{array} \right\} \end{aligned}$$

□

Example 1.11 ("Bernoulli shift"). *See Example 1.6.* $\Omega = [0,1)$, $\mathcal{F} =$ *Borel*, $\mathbf{P} =$ *Lebesgue*, $T\omega := \{2\omega\}$

Theorem 1.5. *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

Proof. (See Example 1.1.) Let $\widetilde{\Omega} = \{0,1\}^{\mathbb{N}}$, $\widetilde{\mathcal{F}} = \dots$, $\widetilde{\mathbf{P}} = \left(\frac{1}{2} : \frac{1}{2}\right)$-Bernoulli, $\widetilde{T} =$ left shift

$$\varphi : \widetilde{\Omega} \to \Omega \qquad \varphi(\widetilde{\omega}) := \sum_{j=0}^{\infty} 2^{-j-1} \widetilde{\omega}_j$$

$$\varphi^{-1} : \Omega \to \widetilde{\Omega} \qquad \varphi^{-1}(\omega)_j := [2^j \omega] \bmod 2$$

Then $(\Omega, \mathcal{F}, \mathbf{P}, T) \stackrel{\varphi:1-1}{\longleftrightarrow} (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$, and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{T})$ is ergodic, according to Example 1.1. □

Alternative proof: by Fourier method (Homework).

Example 1.12 (Algebraic automorphism of the 2-d torus). $\Omega = [0,1) \times [0,1)$, $\mathcal{F} =$ *Borel*, $\mathbf{P} =$ *Lebesgue*,
$T(x,y) := (\{x + 2y\}, \{x + y\})$ *(picture on blackboard)*

Example 1.13 (The "Baker's Transformation"). $\Omega = [0,1) \times [0,1)$, $\mathcal{F} =$ *Borel*, $\mathbf{P} =$ *Lebesgue*,
$T(x,y) := (\{2x\}, \{2x + y/2\})$ *(picture on blackboard)*

In both examples:

Theorem 1.6. *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

Proof.
- Proof 1 Fourier method
- Proof 2 "Markov partition"

□

Example 1.14 (Statistical physics).
- $\Omega =$ *phase space of physical particle system,*
- $\mathcal{F} =$ *Borel,*
- $\mathbf{P} =$ *Liouville measure*
- $=$ *Lebesgue meas. restricted to manifold of conserved quantities,*
- $T_t :=$ *Newtonian dynamical flow*

Theorem 1.7 (Liouville's theorem). *The dynamical flow* $t \mapsto T_t$ *conserves the measure. I.e.* $(\Omega, \mathcal{F}, \mathbf{P}, T_t)$ *is a continuous time dynamical system.*

Ludwig Boltzmann's ergodic hypothesis: In the physically relevant cases, $(\Omega, \mathcal{F}, \mathbf{P}, T_t)$ is ergodic.

Major open question! Answer known in very few cases.

1.2 Koopmanism and von Neumann's (mean, L^2) ergodic theorem

- $(\Omega, \mathcal{F}, \mathbf{P}, T)$: dynamical system,
- $\mathcal{F} \supset \mathcal{I}$: its invariant sigma-algebra,
- $\mathcal{H} := L^2(\Omega, \mathcal{F}, \mathbf{P})$: Hilbert space of square integrable functions,
- $\mathcal{K} := L^2(\Omega, \mathcal{I}, \mathbf{P}) = \{f \in \mathcal{H} : f(T\omega) = f(\omega)\mathbf{P}\text{-a.s.}\}$: subspace of T-invariant L^2-functions.

Two linear operators:

$$\Pi : \mathcal{H} \to \mathcal{K}, \qquad \Pi f(\omega) := \mathbf{E}\left(f \mid \mathcal{I}\right)(\omega)$$
$$U : \mathcal{H} \to \mathcal{H}, \qquad U f(\omega) := f(T\omega)$$

- Π is the orthogonal projection to the subspace $\mathcal{K}$
- U is Koopman's representation of the action T.
- $\mathcal{K} = \mathrm{Ker}(U - I) = \{f \in \mathcal{H} : Uf = f\}$

Lemma 1.1. *U is a (partial) isometry.*

Proof.

$$\begin{aligned}(Uf, Ug) &= \int_\Omega \overline{f(T\omega)} g(T\omega) d\mathbf{P}(\omega) \\ &\overset{(1)}{=} \int_\Omega \overline{f(\omega)} g(\omega) d\mathbf{P}(\omega) = (f, g)\end{aligned}$$

(1) : due to invariance of the measure under the action T.

□

Remark 1.5. *If T is a.s. invertible then U is unitary.*

Theorem 1.8 (von Neumann's mean ergodic theorem). *Let*

- *$\mathcal{H}$: a separable Hilbert space,*
- *$U \in \mathcal{B}(\mathcal{H})$: a (partial) isometry,*
- *$\mathcal{K} := \mathrm{Ker}(U - I)$,*
- *Π: the orthogonal projection to the closed subspace $\mathcal{K}$.*

Then

$$\text{st-}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} U^j = \Pi,$$

That is,

$$\forall f \in \mathcal{H} : \quad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} U^j f - \Pi f \right\| = 0.$$

Corollary 1.1. $(\Omega, \mathcal{F}, \mathbf{P}, T)$: *a dynamical system,* $\mathcal{I}$: *its invariant sigma-algebra.*

If $f \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ *then*

$$\lim_{n\to\infty} \int_\Omega \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j\omega) - \mathbf{E}\left(f \mid \mathcal{I}\right)(\omega)\right|^2 d\mathbf{P}(\omega) = 0.$$

In particular, if $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic then*

$$L^2\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j\omega) = \int_\Omega f d\mathbf{P}.$$

Proof of von Neumann's mean ergodic theorem.

$$\begin{aligned} \mathcal{H} &\overset{(1)}{=} \overline{\mathrm{Ran}(U-I)} \oplus \mathrm{Ker}(U^*-I) \\ &\overset{(2)}{=} \overline{\mathrm{Ran}(U-I)} \oplus \mathrm{Ker}(U-I) \end{aligned}$$

(1) : $\forall A \in \mathcal{B}(\mathcal{H})$: $\quad \mathcal{H} = \overline{\mathrm{Ran}A} \oplus \mathrm{Ker}A^*$

(2) : Since $U \in \mathcal{B}(\mathcal{H})$ is an isometry, $\mathrm{Ker}(U^*-I) = \mathrm{Ker}(U-I)$. (Homework)

For $f \in \mathrm{Ker}(U-I)$:

$$Uf = f = \Pi f \qquad \Rightarrow \qquad \frac{1}{n}\sum_{j=0}^{n-1} U^j f = \Pi f.$$

For $f \in \overline{\mathrm{Ran}(U-I)}$: $(\forall \varepsilon > 0)$ $(\exists g, h \in \mathcal{H})$ such that

$$\|h\| < \varepsilon \quad \text{and} \quad = Ug - g + h.$$

Thus:

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j f = \frac{1}{n}(U^n g - g) + \frac{1}{n}\sum_{j=0}^{n-1} U^j h$$

and hence

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} U^j f \right\| \leq \left(\frac{2}{n} + \varepsilon\right) \|g\|$$

□

1.3 Birkhoff's "individual" (pointwise, almost sure) ergodic theorem

Theorem 1.9 (Birkhoff's individual ergodic theorem). $(\Omega, \mathcal{F}, \mathbf{P}, T)$*: a dynamical system,* $\mathcal{I}$*: its invariant sigma-algebra.*

If $f \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ *then*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j\cdot) \to \mathbf{E}\left(f \mid \mathcal{I}\right)(\cdot)$$

$\mathbf{P}$*-a.s. and in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$*.*

In particular, if $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic then*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j\cdot) \to \int_\Omega f d\mathbf{P}$$

$\mathbf{P}$*-a.s. and in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$*.*

Proof [Birkhoff 1931, Yosida & Kakutani 1939, Garsia 1965]

$$X_j = X_j(\omega) := f(T^j\omega), \qquad X := X_0,$$

$$S_k = S_k(\omega) := \sum_{j=0}^{k-1} X_j(\omega), \qquad S_0 = 0,$$

$$M_k = M_k(\omega) := \max\{S_j(\omega) : j = 0, 1, \dots, k\}, \qquad M_0 = 0.$$

Lemma 1.2 (The maximal ergodic lemma).

$$\mathbf{E}\left(X 1\!\!1_{\{M_k>0\}}\right) \geq 0$$

Explicitly spelled out:

$$\int_\Omega f(\omega) 1\!\!1_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega) \geq 0$$

Mind the strict inequality: $M_k > 0$!

Proof of the maximal lemma (Garsia 1965).

$$\begin{aligned} X(\omega) &\overset{(1)}{=} \max\{S_j(\omega) : j = 1, \dots, k+1\} - \max\{S_j(T\omega) : j = 0, \dots, k\} \\ &\geq \max\{S_j(\omega) : j = 1, \dots, k\} - \max\{S_j(T\omega) : j = 0, \dots, k\} \\ &= \max\{S_j(\omega) : j = 1, \dots, k\} - M_k(T\omega) \end{aligned}$$

(1) : Since $S_{j+1}(\omega) = X(\omega) + S_j(T\omega), \quad j = 0, 1, \dots$.

Hence

$$\begin{aligned} &\int_\Omega X(\omega) \mathbb{1}_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega) \\ &\qquad \geq \int_\Omega \left(\max\{S_j(\omega) : j = 1, \dots, k\} - M_k(T\omega)\right) \mathbb{1}_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega) \\ &\qquad \overset{(2)}{=} \int_\Omega \big(M_k(\omega) - M_k(T\omega)\big) \mathbb{1}_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega) \\ &\qquad \geq \int_\Omega \big(M_k(\omega) - M_k(T\omega)\big) d\mathbf{P}(\omega) \overset{(3)}{=} 0. \end{aligned}$$

(2) : Here we use the strict inequality $M_k > 0$.

(3) : Due to invariance of the measure under the action T.

□

Proof of Birkhoff's theorem. Without loss of generality assume $\mathbf{E}\left(f \mid \mathcal{I}\right) = 0$. Fix $\varepsilon > 0$ and define

$$L(\omega) := \limsup_{n\to\infty} \frac{S_n(\omega)}{n}, \qquad D^\varepsilon := \{\omega : L(\omega) > \varepsilon\} \in \mathcal{I},$$

$$X^\varepsilon(\omega) := \big(X(\omega) - \varepsilon\big) \mathbb{1}_{D^\varepsilon}(\omega), \qquad S^\varepsilon_k(\omega) := \sum_{j=0}^{k-1} X^\varepsilon_j(\omega),$$

$$M^\varepsilon_k(\omega) := \max\{S^\varepsilon_j(\omega) : j = 0, \dots, k\}, \qquad F^\varepsilon := \cup_k \{\omega : M^\varepsilon_k(\omega) > 0\}.$$

Note that

$$F^\varepsilon = \{\omega : \sup_k M^\varepsilon_k(\omega) > 0\} = \{\omega : \sup_k S^\varepsilon_k(\omega) > 0\} = D^\varepsilon$$

$$
\begin{aligned}
0 &\overset{(1)}{\leq} \mathbf{E}\left(X^{\varepsilon}1\!\!1_{\{M_n^{\varepsilon}>0\}}\right) \overset{(2)}{\to} \mathbf{E}\left(X^{\varepsilon}1\!\!1_{F^{\varepsilon}}\right) \\
&\overset{(3)}{=} \mathbf{E}\left(X^{\varepsilon}1\!\!1_{D^{\varepsilon}}\right) \overset{(4)}{=} \mathbf{E}\left((X-\varepsilon)1\!\!1_{D^{\varepsilon}}\right) \overset{(5)}{=} -\varepsilon\mathbf{P}\left(D^{\varepsilon}\right)
\end{aligned}
$$

(1) : due to the maximal lemma

(2) : dominated convergence

(3) : since $F^{\varepsilon} = D^{\varepsilon}$

(4) : by definition of X^{ε}

(5) : since $D^{\varepsilon} \in \mathcal{I}$ and $\mathbf{E}\left(X \,\middle|\, \mathcal{I}\right) = 0$.

It follows that $\forall \varepsilon > 0 : \ \mathbf{P}\left(D^{\varepsilon}\right) = 0$, and

$$
\mathbf{P}\left(L > 0\right) = \mathbf{P}\left(\cup_{\varepsilon>0} D^{\varepsilon}\right) = \lim_{\varepsilon\to 0} \mathbf{P}\left(D^{\varepsilon}\right) = 0.
$$

□

1.4 Back to the examples

Example 1.15 (I.i.d. sequence). *See Example 1.1.* X_j, *i.i.d.,* $\mathbf{E}\left(|X_j|\right) < \infty$.

$$
\frac{1}{n}\sum_{j=0}^{n-1} X_j \to \mathbf{E}\left(X_j\right).
$$

Laws of large numbers.

Example 1.16 (Factors). *See Example 1.2 and Example 1.3. Laws of large numbers for factors of i.i.d. sequences.*

Example 1.17 (Stationary denumerable Markov chains). *See Example 1.4.*

ξ_j: *stationary MC on* $S = \cup_m S^{(m)}$, *(*$S^{(m)}$ *irred. comp.)*

$$f : S \to \mathbb{R} : \qquad \sum_{\alpha \in S} \pi(\alpha) \, | f(\alpha) \, | < \infty.$$

$$\frac{1}{n} \sum_{j=0}^{n-1} f(\xi_j) \to \sum_m \mathbb{1}_{\{\xi_0 \in S^{(m)}\}} \frac{\sum_{\alpha \in S^{(m)}} \pi(\alpha) f(\alpha)}{\sum_{\alpha \in S^{(m)}} \pi(\alpha)}.$$

Law of large numbers for MC.

Example 1.18 (Rotations of the circle). *See Example 1.5.* $\theta \notin \mathbb{Q}$, $f \in L^1([0,1), \mathcal{B}, d\omega)$*:*

$$\frac{1}{n} \sum_{j=0}^{n-1} f(\cdot + j\theta) \to \int_0^1 f(\omega) d\omega, \qquad a.s. \text{ and in } L^1.$$

Remark 1.6. *For* $f := \mathbb{1}_{[a,b)}$ *stronger:*

$$\forall \omega \in [0,1) : \qquad \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[a,b)}(\omega + j\theta) \to b - a.$$

Proof: Homework.

Consequence 1.1. *Fix* $k \in \{1, 2, \ldots, 9\}$*. Then*

$$\frac{\#\{m < n : 2^m = k \cdots \text{ in dec.}\}}{n} \to \frac{\log(k+1) - \log k}{\log 10}$$

Proof. Let $\theta := \frac{\log 2}{\log 10} \notin \mathbb{Q}$.

$$\{2^m = k \cdots \text{in dec.}\} \Leftrightarrow \{\{m\theta\} \in A_k := [\log k / \log 10, \log(k+1)/\log 10)\}$$

□

Example 1.19 (Bernoulli shift). *See Example 1.6.*

$$\omega \in [0,1), \qquad \textit{binary expansion:} \qquad \omega = \sum_{j=1}^{\infty} \omega_j 2^{-j}$$

Theorem 1.10. *For Lebesgue-a.e.* $\omega \in [0,1)$ *any fixed* $\{0,1\}$ *string* $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ *occurs with its natural proper density* 2^{-k}.

I.e. "Almost all real numbers are normal."

Example 1.20 (Statistical physics).

$$\textit{Ergodicity}$$
$$\Updownarrow$$
$$\left\{ \textit{time averages} = \textit{phase space averages} \right\}$$

At the heart of statistical physics.

Chapter 2

Convergence in distribution, weak convergence

2.1 Convergence in distribution, basics

- (S, d) complete, separable metric space,
- $\mathcal{S}$ its Borel-sigma-algebra,
- e.g. $\mathbb{R}$, $\mathbb{R}^n$ with Euclidean distance,
- $C([0,1])$, $C([0,\infty))$ with sup-norm distance.

Definition 2.1. *A probability measure ν on $(S, \mathcal{S})$ is regular if* $(\forall A \in \mathcal{S})$

$$\begin{aligned} \nu(A) &= \sup\{\nu(K) : K \subseteq A,\ K \text{ compact}\} \\ &= \inf\{\nu(O) : A \subseteq O,\ O \text{ open}\} \end{aligned}$$

All measures considered will be assumed regular.
μ_n, $n = 1, 2, \ldots$ and μ regular probability measures on $(S, \mathcal{S})$.
Y_n, $n = 1, 2, \ldots$ and Y S-valued r.v. with distribution

$$\mathbf{P}(Y_n \in A) = \mu_n(A), \quad \mathbf{P}(Y \in A) = \mu(A), \qquad A \in \mathcal{S}$$

not necessarily jointly defined.

Definition 2.2 (Weak convergence of probability measures). $\mu_n \Rightarrow \mu$, *or* $Y_n \Rightarrow Y$, *iff* $\forall f : S \to \mathbb{R}$ *continuous and bounded*

$$\lim_{n\to\infty} \int_S f d\mu_n = \int_S f d\mu, \quad \text{or} \quad \lim_{n\to\infty} \mathbf{E}\left(f(Y_n)\right) = \mathbf{E}\left(f(Y)\right).$$

Theorem 2.1 (Equiv. characterizations, "portmanteau thm"). *(a)≡(b)≡(c)≡(d)*

(a) $\mu_n \Rightarrow \mu$.

(b) $(\forall A \in \mathcal{S})$, A *open:* $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$.

(c) $(\forall A \in \mathcal{S})$, A *closed:* $\limsup_{n\to\infty} \mu_n(A) \leq \mu(A)$.

(d) $(\forall A \in \mathcal{S})$, *such that* $\mu(\partial A) = 0$*:* $\lim_{n\to\infty} \mu_n(A) = \mu(A)$.

Proof. Probability 2. □

2.1.1 The special case of $\mathbb{R}$ (or $\mathbb{R}^d$)

The distribution function helps:

$$\begin{aligned} F_n(x) &:= \mathbf{P}\left(Y_n < x\right) = \mu_n\big((-\infty, x)\big), \\ F(x) &:= \mathbf{P}\left(Y < x\right) = \mu\big((-\infty, x)\big). \end{aligned}$$

Theorem 2.2. $\mu_n \Rightarrow \mu$ *(also denoted* $F_n \Rightarrow F$*) iff*

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{at all points of continuity of } F.$$

Proof. Probability 2. □

2.1.2 Examples for weak convergence

Example 2.1. *Convergence in probability (*Probability 2, Analysis*)— this is NOT the typical case:* $(\Omega, \mathcal{F}, \mathbf{P})$

$$Y_n, Y : \Omega \to \mathbb{R} \quad \textit{defined on the same probab. sp.}, \qquad Y_n \xrightarrow{\mathbf{P}} Y$$

Example 2.2. *Poisson approximation of binomial (*Probability 1*):*

$$Y_n \sim BIN(p_n, n), \quad \lim_{n\to\infty} np_n = \lambda \in (0, \infty), \quad Y \sim POI(\lambda).$$

Example 2.3. *De Moivre's CLT (*Probability 1*):*

$$\widetilde{Y}_n \sim BIN(p, n), \qquad Y_n := \frac{\widetilde{Y}_n - pn}{\sqrt{p(1-p)n}}, \qquad Y \sim N(0, 1).$$

Example 2.4. *De Moivre's-type CLT for gamma-distributions (*Probability 2*):*

$$\widetilde{Y}_n \sim GAM(\lambda, n), \qquad Y_n := \frac{\widetilde{Y}_n - \lambda^{-1} n}{\sqrt{\lambda^{-1} n}}, \qquad Y \sim N(0, 1).$$

Example 2.5. *General CLT for sums of i.i.d. r.v.-s (*Probability 2*) — the typical case:*

$$X_n \text{ i.i.d. r.v.-s}, \qquad m := \mathbf{E}(X_j), \qquad \sigma^2 := \mathbf{Var}(X_j),$$
$$Y_n := \frac{\sum_{j=1}^n (X_j - m)}{\sigma\sqrt{n}}, \qquad Y \sim N(0, 1)$$

2.1.3 Tightness

Definition 2.3. *The sequence of probability measures* μ_n *on* $(S.\mathcal{S})$, *or the sequence of* S*-valued random variables* Y_n, *is tight, if* $(\forall\varepsilon > 0)$ $(\exists K \Subset S)$ *such that*

$$(\forall n): \qquad \mu_n\big(S \setminus K\big) < \varepsilon, \quad \text{or} \quad \mathbf{P}\left(Y_n \notin K\right) < \varepsilon.$$

In the $S = \mathbb{R}$ *case* $(\forall\varepsilon > 0)$ $(\exists K < \infty)$ *such that*

$$(\forall n): \qquad \mu_n\big((-\infty, -K) \cup (K, \infty)\big) < \varepsilon, \quad \text{or} \quad \mathbf{P}\left(|Y_n| > K\right) < \varepsilon.$$

Proposition 2.1. *If* $\mu_n \Rightarrow \mu$ *then the sequence* μ_n *is tight.*

Proof. Easy, if S is locally compact!

Choose

$$\widetilde{K} \Subset K \Subset S \quad \text{s.t.} \quad \mu\big(S \setminus \widetilde{K}\big) < \varepsilon/2$$

and

$$f: S \to [0,1] \quad \text{cont., s.t.} \quad f\restriction_{\widetilde{K}} = 0, \quad f\restriction_{S\setminus K} = 1.$$

Then

$$\begin{array}{ccccc} \mu_n\big(S \setminus K\big) \leq & \int_S f d\mu_n & \leq \mu_n\big(S \setminus \widetilde{K}\big) \\ & \downarrow & \\ \mu\big(S \setminus K\big) \leq & \int_S f d\mu & \leq \mu\big(S \setminus \widetilde{K}\big) < \varepsilon/2. \end{array}$$

Hence, $(\exists n_0 < \infty)$ such that $(\forall n \geq n_0): \mu_n\big(S \setminus K\big) < \varepsilon$. □

Theorem 2.3 (Helly's theorem). *Let* $\{\mu_n/F_n/Y_n\}$, $n = 1, 2, \ldots,$ *be a tight sequence of {probability measures / probability distribution functions / random variables} on* $\mathbb{R}$. *Then one can extract a weakly convergent subsequence* $\{\mu_{n_k}/F_{n_k}/Y_{n_k}\}$, $k = 1, 2, \ldots$ *:*

$$\{\quad \mu_{n_k} \Rightarrow \mu \quad / \quad F_{n_k} \Rightarrow F \quad / \quad Y_{n_k} \Rightarrow Y \quad\} \qquad \text{as} \qquad k \to \infty.$$

Theorem 2.4 (Prohorov's theorem). *Let* $\{\mu_n \;/\; Y_n\}$, $n = 1, 2, \dots$, *be a tight sequence of* {*probability measures / random variables*} *on the complete separable metric space* S. *Then one can extract a weakly convergent subsequence* $\{\mu_{n_k} \;/\; Y_{n_k}\}$, $k = 1, 2, \dots$ *:*

$$\{ \quad \mu_{n_k} \Rightarrow \mu \quad / \quad Y_{n_k} \Rightarrow Y \quad \} \qquad \text{as} \qquad k \to \infty.$$

For proof of both Thms see: *Probability 2.*

2.2 Methods for proving weak convergence

General scheme

(1) prove tightness

(2) prove uniqueness of possible limits

(3) identify the limit

Methods

(A) With bare hands (e.g. De Moivre, Poisson, maxima of i.i.d.)

(B) Method of moments

(C) Method of characteristic functions (e.g. Markov–Lévy CLT)

(D) Coupling

(E) Mixed methods

2.3 With bare hands

2.3.1 Arcsine laws and related stuff

X_n simple symmetric random walk on $\mathbb{Z}$ $(d = 1!)$:

$$X_0 = 0, \qquad \mathbf{P}\left(X_{n+1} = i \pm 1 \,\middle|\, X_n = i\right) = \frac{1}{2}.$$

Some relevant random variables:

The maximum:	$M_n := \max\{X_j : j \in [0, n]\}$,
First hitting of $r \in \mathbb{Z}_+$:	$T_r := \inf\{n > 0 : X_n = r\}$,
Return times $k \in \mathbb{N}$: $R_0 = 0$,	$R_{k+1} := \inf\{n > R_k : X_n = 0\}$,
Local time at $0 \in \mathbb{Z}$:	$L_n := \#\{j \in (0, n] : X_j = 0\}$,
Last visit to $0 \in \mathbb{Z}$:	$\lambda_n := \max\{j \in (0, n] : X_j = 0\}$,
Time spent on $\mathbb{Z}_+$:	$\pi_n := \#\{j \in (0, n] : \frac{X_{j-1} + X_j}{2} > 0\}$.

Theorem 2.5 (Limit theorem for the maximum). *(i) Discrete,microscopic version: $0 \le r \le n$ fixed:*

$$\mathbf{P}(M_n = r) = \mathbf{P}(X_n = r) + \mathbf{P}(X_n = r + 1).$$

(ii) Local limit theorem: $0 \le u$ fixed, $1 \ll n$:

$$n^{1/2}\mathbf{P}\left(M_n = [n^{1/2}u]\right) = \sqrt{\frac{2}{\pi}}e^{-u^2/2}\mathbb{1}_{u>0} + \mathcal{O}(n^{-1/2})$$

(iii) Global (integrated) limit theorem: $0 \le x$ fixed:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\left(n^{-1/2}M_n < x\right) &= \mathbb{1}_{x>0}\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}du \\ &= \mathbb{1}_{x>0}(2\Phi(x) - 1).\end{aligned}$$

Proof of part (i).

$$\begin{aligned}\mathbf{P}(M_n \ge r) &= \mathbf{P}(M_n \ge r, X_n \ne r) + \mathbf{P}(M_n \ge r, X_n = r) \\ &\overset{*}{=} 2\mathbf{P}(M_n \ge r, X_n > r) + \mathbf{P}(M_n \ge r, X_n = r) \\ &= 2\mathbf{P}(X_n \ge r) - \mathbf{P}(X_n = r).\end{aligned}$$

* due to the reflection principle.

$$\begin{aligned}
\mathbf{P}(M_n = r) &= \mathbf{P}(M_n \geq r) - \mathbf{P}(M_n \geq r+1) \\
&= 2\mathbf{P}(X_n \geq r) - 2\mathbf{P}(X_n \geq r+1) - \\
&\qquad -\mathbf{P}(X_n = r) + \mathbf{P}(X_n = r+1) \\
&= \mathbf{P}(X_n = r) + \mathbf{P}(X_n = r+1)
\end{aligned}$$

□

Proof of parts (ii) and (iii).

$$\begin{aligned}
\mathbf{P}\left(M_n = [\sqrt{n}u]\right) &= \mathbf{P}\left(X_n = [\sqrt{n}u]\right) + \mathbf{P}\left(X_n = [\sqrt{n}u] + 1\right) \\
&\stackrel{**}{=} n^{-1/2}\sqrt{\frac{2}{\pi}}e^{-u^2/2} + \mathcal{O}(n^{-1})
\end{aligned}$$

** due to De Moivre.

(iii) Integrated version follows from local version + Fatou + Riemannian integration.

□

Theorem 2.6 (Limit theorem for the hitting times). *(i) Discrete, microscopic version:* $0 < r \leq n$ *fixed:*

$$\mathbf{P}(T_r = n) = \frac{r}{n}\binom{n}{(n+r)/2}2^{-n}$$

(ii) Local limit theorem: $0 < s$ *fixed,* $1 \ll r$*:*

$$r^2\mathbf{P}\left(T_r = [r^2 s]\right) = \sqrt{\frac{2}{\pi}}s^{-3/2}e^{-1/(2s)}\mathbb{1}_{s>0} + \mathcal{O}(r^{-1}).$$

(iii) Global (integrated) limit theorem: $0 < t$ *fixed:*

$$\begin{aligned}
\lim_{r\to\infty}\mathbf{P}\left(r^{-2}T_r < t\right) &= \mathbb{1}_{t>0}\frac{1}{\sqrt{2\pi}}\int_0^t s^{-3/2}e^{-1/(2s)}ds \\
&= \mathbb{1}_{t>0}\sqrt{\frac{2}{\pi}}\int_{1/\sqrt{t}}^{\infty} e^{-u^2/2}du.
\end{aligned}$$

Proof of part (i).

$$
\begin{aligned}
\mathbf{P}(T_r = n) &= \frac{1}{2}\mathbf{P}\left(\{\max_{j\le n-2} X_j \le r-1\} \wedge \{X_{n-1} = r-1\}\right) \\
&= \frac{1}{2}\mathbf{P}(X_{n-1} = r-1) - \\
&\qquad -\frac{1}{2}\mathbf{P}\left(\{\max_{j\le n-2} X_j \ge r\} \wedge \{X_{n-1} = r-1\}\right) \\
&\overset{*}{=} \frac{1}{2}\mathbf{P}(X_{n-1} = r-1) - \frac{1}{2}\mathbf{P}(X_{n-1} = r+1) \\
&= \frac{r}{n}\binom{n}{(n+r)/2}2^{-n}
\end{aligned}
$$

* due to the reflection principle.

□

Proof of parts (ii) and (iii).

$$
\begin{aligned}
\mathbf{P}\left(T_r = [r^2 s]\right) &= \frac{r}{[r^2 s]}\binom{[r^2 s]}{([r^2 s]+r)/2}2^{-[r^2 s]} \\
&\overset{**}{=} r^{-2}\frac{2}{\sqrt{2\pi}}s^{-3/2}e^{-1/(2s)} + \mathcal{O}(r^{-3})
\end{aligned}
$$

** due to Stirling.

(iii) Integrated version: local version + Fatou + Riemannian integration.

□

Theorem 2.7 (Limit theorem for the return times). *(i) Discrete, microscopic version:* $0 < k \leq n$ *fixed:*

$$\mathbf{P}\left(R_k = k + n\right) = \frac{k}{n}\binom{n}{(n+k)/2}2^{-n}$$

(ii) Local limit theorem: $0 < s$ *fixed:*

$$k^2\mathbf{P}\left(R_k = [k^2 s]\right) = \frac{1}{\sqrt{2\pi}}s^{-3/2}e^{-1/(2s)}1\!\!1_{s>0} + \mathcal{O}(k^{-1}).$$

(iii) Global (integrated) version: $0 < t$ *fixed:*

$$\begin{aligned}\lim_{k\to\infty}\mathbf{P}\left(k^{-2}R_k < t\right) &= 1\!\!1_{t>0}\frac{1}{\sqrt{2\pi}}\int_0^t s^{-3/2}e^{-1/(2s)}ds.\\ &= 1\!\!1_{t>0}\sqrt{\frac{2}{\pi}}\int_{1/\sqrt{t}}^\infty e^{-u^2/2}du.\end{aligned}$$

Proof.

$$R_k \overset{\text{law}}{=} T_k + k.$$

□

Remarks on the last two limit theorems

Remark 2.1 (I.i.d. sums).

$$T_r = \xi_1 + \xi_2 + \cdots + \xi_r, \quad R_k = \zeta_1 + \zeta_2 + \cdots + \zeta_k,$$

where ξ_i, $i = 1, 2, \ldots$ *and* ζ_i, $i = 1, 2, \ldots$ *are sequences of i.i.d. r.v.-s with*

$$\xi_i \overset{law}{=} T_1, \qquad \zeta_i \overset{law}{=} R_1 \overset{law}{=} T_1 + 1.$$

Remark 2.2 (Stability).

$$f_1(s) := \frac{1}{\sqrt{2\pi}} s^{-3/2} e^{-1/(2s)} \mathbb{1}_{s>0}, \qquad f_a(s) := a f_1(as), \quad a > 0.$$

Then

$$f_a * f_b = f_{(\sqrt{a}+\sqrt{b})^2}$$

Homework.

Theorem 2.8 (Limit theorem for the local time at zero). *Global (integrated) version:*

$$\lim_{n\to\infty} \mathbf{P}\left(n^{-1/2} L_n < t\right) = \mathbb{1}_{t>0} \sqrt{\frac{2}{\pi}} \int_0^t e^{-u^2/2} du.$$

Proof.

$$\{L_n < k\} = \{R_k > n\}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}\left(L_n < n^{1/2} t\right) &= \lim_{n\to\infty} \mathbf{P}\left(R_{n^{1/2}t} > n\right) = \lim_{m\to\infty} \mathbf{P}\left(R_m > m^2/t^2\right) \\ &= \sqrt{\frac{2}{\pi}} \int_0^t e^{-u^2/2} du. \end{aligned}$$

□

Remark 2.3. *Note that*

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}\left(n^{-1/2} |X_n| < u\right) &= \lim_{n\to\infty} \mathbf{P}\left(n^{-1/2} L_n < u\right) \\ &= \lim_{n\to\infty} \mathbf{P}\left(n^{-1/2} M_n < u\right) \end{aligned}$$

For a simple symmetric random walk X_n (on $\mathbb{Z}$) denote

$$u(n) := \mathbf{P}(X_n = 0) = \binom{n}{n/2} 2^{-n}$$

$$f(n) := \mathbf{P}(\min\{m \geq 1 : X_m = X_0\} = n)$$

Recall the identity:

$$u(n) = \sum_{m=0}^{n} f(m)u(n-m).$$

Theorem 2.9 (Paul Lévy's arcsine theorem). *(i) Discrete, microscopic version:* $0 \leq k \leq n$*:*

$$\mathbf{P}(\lambda_{2n+1} = 2k) \stackrel{\checkmark}{=} \mathbf{P}(\lambda_{2n} = 2k) = u(2k)u(2n-2k),$$
$$\mathbf{P}(\pi_{2n+1} \in \{2k, 2k+1\}) \stackrel{\checkmark}{=} \mathbf{P}(\pi_{2n} = 2k) = u(2k)u(2n-2k),$$
$$\Big(\mathbf{P}(\lambda_{2n} = 2k+1) \stackrel{\checkmark}{=} \mathbf{P}(\lambda_{2n+1} = 2k+1) \stackrel{\checkmark}{=} \mathbf{P}(\pi_{2n} = 2k+1) \stackrel{\checkmark}{=} 0\Big)$$

(ii) Local limit theorem: $y \in (0,1)$ *fixed* $1 \ll n$*:*

$$n\mathbf{P}(\lambda_{2n} = 2[ny]) = n\mathbf{P}(\pi_{2n} = 2[ny]) = \frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}} + \mathcal{O}(n^{-1/2})$$

(iii) Global (integrated) limit theorem: $x \in (0,1)$ *fixed*

$$\lim_{n\to\infty} \mathbf{P}\left(n^{-1}\lambda_n < x\right) = \lim_{n\to\infty} \mathbf{P}\left(n^{-1}\pi_n < x\right) = 1\!\!1_{0<x<1}\frac{2}{\pi}\arcsin\sqrt{x}.$$

Lemma 2.1.

$$\mathbf{P}(X_j \neq 0,\ j = 1, 2, \ldots, 2n) = \mathbf{P}(X_{2n} = 0) =: u(2n).$$

Proof of Lemma 2.1.

$$\begin{aligned}
&\mathbf{P}\left(X_j \neq 0,\ j = 1,2,\dots,2n\right) \\
&\quad = 2\mathbf{P}\left(X_j > 0,\ j = 1,2,\dots,2n\right) \\
&\quad = 2\sum_{r=1}^{\infty} \mathbf{P}\left(\{X_j > 0,\ j = 1,2,\dots,2n-1\} \wedge \{X_{2n} = 2r\}\right) \\
&\quad \overset{*}{=} 2\sum_{r=1}^{\infty} \frac{1}{2}\big(\mathbf{P}\left(X_{2n-1} = 2r-1\right) - \mathbf{P}\left(X_{2n-1} = 2r+1\right)\big) \\
&\quad = \mathbf{P}\left(X_{2n-1} = 1\right) = \mathbf{P}\left(X_{2n} = 0\right).
\end{aligned}$$

* due to the reflection principle.

□

Proof of Theorem 2.9. (i) For λ_n:

$$\begin{aligned}
\mathbf{P}\left(\lambda_{2n} = 2k\right) &= \mathbf{P}\left(\{X_{2k=0}\} \wedge \{X_j \neq 0,\ j = 2k+1,\dots,2n\}\right) \\
&= \mathbf{P}\left(X_{2k=0}\right)\mathbf{P}\left(X_j \neq 0,\ j = 1,\dots,2n-2k)\}\right) \\
&= u(2k)u(2n-2k).
\end{aligned}$$

For π_n by induction. Note that

$$\mathbf{P}\left(\pi_{2n} = 2k\right) = \mathbf{P}\left(\pi_{2n} = 2n-2k\right).$$

For $k = 0$ or $k = n$:

$$\begin{aligned}
\mathbf{P}\left(\pi_{2n} = 0\right) &= \mathbf{P}\left(X_j \geq 0,\ j = 1,2,\dots,2n\right) \\
&= \mathbf{P}\left(X_j \geq 0,\ j = 1,2,\dots,2n-1\right) \\
&= 2\mathbf{P}\left(X_j > 0,\ j = 1,2,\dots,2n\right) = u(2n)u(0)
\end{aligned}$$

Denote

$$b(2n,2k) := \mathbf{P}\left(\pi_{2n} = 2k\right) = b(2n,2n-2k)$$

For $1 \leq k \leq n$ there is a first excursion to the left or to the right:

$$b(2n,2k) = \frac{1}{2}\sum_{r=1}^{k} f(2r)b(2n-2r,2k-2r) + \frac{1}{2}\sum_{r=1}^{n-k} f(2r)b(2n-2r,2k)$$

By the induction assumption:

$$\begin{aligned} b(2n,2k) &= \frac{1}{2}u(2n-2k)\sum_{r=1}^{k} f(2r)u(2k-2r) + \\ &\quad +\frac{1}{2}u(2k)\sum_{r=1}^{n-k} f(2r)u(2n-2k-2r) \\ &= \frac{1}{2}u(2n-2k)u(2k)+\frac{1}{2}u(2k)u(2n-2k) = u(2k)u(2n-2k) \end{aligned}$$

(ii)

$$u(2[ny])u(2[n(1-y)]) \overset{**}{=} n^{-1}\frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}} + \mathcal{O}(n^{-3/2})$$

** due to Stirling.

(iii) Integrated version: local version + Fatou + Riemannian integration.

□

Chapter 3

The method of moments and the method of characteristic functions

- *Recall everything you learnt about characteristic functions.*
- *Probability II.*

3.1 The method of moments

Let X be a random variable, its absolute moments and its moments are assumed finite:

$$A_k := \mathbf{E}\left(|X|^k\right) < \infty, \qquad M_k := \mathbf{E}\left(X^k\right)$$

Remark 3.1. *In order that the sequences A_k and M_k be the sequences of (absolute) moments of a random variable X it must satisfy an infinite set of (Jensen-type) inequalities: in particular, if $k_1 + \cdots + k_m = k$, respectively, if $k_1 + \cdots + k_m = 2k$ then*

$$\prod_{j=1}^{m} A_{k_j} \leq A_k, \qquad \prod_{j=1}^{m} |M_{k_j}| \leq M_{2k},$$

The "Moment problem": Given a sequence of moments M_k, does it determine uniquely the distribution of a random variable?

Theorem 3.1. *If M_k is a sequence of moments such that*

$$\limsup_{k\to\infty}\left(\frac{|M_k|}{k!}\right)^{1/k} := R^{-1} < \infty$$

then it determines a unique random variable X (or: probability distribution) such that $M_k = \mathbf{E}\left(X^k\right)$.

Proof. The power series of the characteristic function

$$\sum_{k=0}^{\infty}\frac{M_k}{k!}(iu)^k$$

will have radius of convergence $R > 0$, and thus it will be uniquely determined. □

Example 3.1. *Compute all moments of all remarkable distributions. E.g.*

$$\begin{aligned} X \sim EXP(\lambda):& \qquad M_k = A_k = \lambda^{-k}k! \\ X \sim N(0,\sigma):& \qquad A_{2k} = \sigma^{2k}\frac{2k!}{2^k k!} = M_{2k}, \\ & \qquad A_{2k+1} = \sigma^{2k+1}\sqrt{\frac{2}{\pi}}2^k k!, \quad M_{2k+1} = 0 \end{aligned}$$

Counterexample 3.1. *The* log-normal *distribution (HW!).*

3.1.1 Weak limit from convergence of moments

Theorem 3.2. *Let Z_n be a sequence of random variables which have all moments finite and denote*

$$M_{n,k} := \mathbf{E}\left(Z_n^k\right).$$

If $(\forall k)$ the limit $\lim_{n\to\infty} M_{n,k} =: M_k$ exists and the sequence of moments M_k determines uniquely *a distribution/random variable Z, then $Z_n \Rightarrow Z$.*

Remark 3.2. *The sequence M_k is a sequence of moments.*

Proof. (i) Tightness:

$$\mathbf{P}(|Z_n| > K) \leq \frac{M_{n,2}}{K^2}. \leq \frac{\sup_n M_{n,2}}{K^2}.$$

(ii) Identification of the limit: Assume $Z_{n'} \Rightarrow \widetilde{Z}$. For $K < \infty$ let $\varphi_K : \mathbb{R} \to \mathbb{R}$,

$$\varphi_K(x) := x\mathbb{1}_{|x|\leq K} + \mathrm{sgn}(x)K\mathbb{1}_{|x|>K}.$$

Then

$$\begin{aligned}
\mathbf{E}\left(\widetilde{Z}^k\right) &= \lim_{K\to\infty} \mathbf{E}\left(\varphi_K(\widetilde{Z})^k\right) \\
&= \lim_{K\to\infty}\lim_{n'\to\infty} \mathbf{E}\left(\varphi_K(Z_{n'})^k\right) \qquad \text{(due to weak cvg.)} \\
&= \lim_{K\to\infty}\lim_{n'\to\infty} \left(\mathbf{E}\left(Z_{n'}^k\right) - \mathbf{E}\left(Z_{n'}^k - \varphi_K(Z_{n'})^k\right)\right) \\
&= \lim_{n'\to\infty} M_{n',k} - \lim_{K\to\infty}\lim_{n'\to\infty} \mathbf{E}\left(Z_{n'}^k - \varphi_K(Z_{n'})^k\right)
\end{aligned}$$

But:

$$\begin{aligned}
\left|\mathbf{E}\left(Z_{n'}^k - \varphi_K(Z_{n'})^k\right)\right| &\leq \mathbf{E}\left(\left|Z_{n'}\right|^k \mathbb{1}_{|Z'_n|>K}\right) \\
&\stackrel{(1)}{\leq} \sqrt{M_{n',2k}}\,\sqrt{\mathbf{P}(|Z'_n| > K)} \\
&\stackrel{(2)}{\leq} \frac{\sqrt{M_{n',2k}}\,\sqrt{M_{n',2}}}{K}
\end{aligned}$$

(1) : due to Schwarz's inequality

(2) : due to Markov's inequality

Altogether:

$$\mathbf{E}\left(\widetilde{Z}^k\right) = M_k.$$

□

3.1.2 Appl 1: CLT with the method of moments

Sheds light on the *combinatorial aspects* of the CLT. Let ξ_j be i.i.d. with all moments finite, $\mathbf{E}\left(\xi_j^k\right) =: m_k, \quad m_1 = 0, \quad m_2 =: \sigma^2$,

$$Z_n := \frac{\xi_1 + \cdots + \xi_n}{\sqrt{n}}.$$

Then, with fixed k:

$$\begin{aligned} \mathbf{E}\left(Z_n^{2k}\right) &= n^{-k}\binom{n}{k}\sigma^{2k}\frac{2k!}{2^k} + o(1) \to \sigma^{2k}\frac{2k!}{2^k k!}, \\ \mathbf{E}\left(Z_n^{2k+1}\right) &= o(1) \to 0, \end{aligned}$$

as $n \to \infty$ (with k fixed).

3.2 The method of characteristic functions (Repeat from Probability II.)

Theorem 3.3. *Let Z_n be a sequence of random variables and $\varphi_n : \mathbb{R} \to \mathbb{R}$ their characteristic functions,*

$$\varphi_n(u) := \mathbf{E}\left(\exp(iuZ_n)\right).$$

If

$$(\forall u \in \mathbb{R}) : \quad \lim_{n\to\infty} \varphi_n(u) = \varphi(u) \qquad \textit{(pointwise!)}$$

and $u \mapsto \varphi(u)$ is continuous at $u = 0$, then φ is characteristic function of a random variable Z and $Z_n \Rightarrow Z$.

For proving tightness:

Lemma 3.1 (Paul Lévy). *Let Y be a random variable and $\psi(u) := \mathbf{E}\left(\exp(iuY)\right)$ its characteristic function. Then for any $K < \infty$*

$$\mathbf{P}\left(|Y| > K\right) \leq \frac{K}{2}\int_{-2/K}^{2/K} (1 - \psi(u))du.$$

Proof of Lemma 3.1.

$$\begin{aligned}
\frac{K}{2}\int_{-2/K}^{2/K}(1-\psi(u))du &= \frac{K}{2}\int_{-2/K}^{2/K}\mathbf{E}\left(1-e^{iuY}\right)du \\
&\overset{(1)}{=} 2\mathbf{E}\left(1-\frac{\sin(2Y/K)}{2Y/K}\right) \\
&\overset{(2)}{\geq} 2\mathbf{E}\left(\left(1-\frac{\sin(2Y/K)}{2Y/K}\right)1\!\!1_{|Y|>K}\right) \\
&\overset{(3)}{\geq} 2\mathbf{E}\left(\left(1-\frac{K}{2|Y|}\right)1\!\!1_{|Y|>K}\right) \\
&\geq \mathbf{P}\left(|Y|>K\right).
\end{aligned}$$

(1) : Fubini,

(2) : $|\sin\alpha/\alpha| \leq 1$,

(3) : $\sin\alpha/\alpha \leq 1/|\alpha|$.

□

Proof of Theorem 3.3. (1) Tightness:

From continuity of $u \mapsto \varphi(u)$ at $u = 0$:

$$(\exists K<\infty): \qquad \frac{K}{2}\int_{-2/K}^{2/K}(1-\varphi(u))du < \frac{\varepsilon}{2}.$$

From pointwise convergence (and uniform boundedness of φ_n)

$$(\exists n_0<\infty): \qquad (\forall n \geq n_0): \quad \frac{K}{2}\int_{-2/K}^{2/K}(1-\varphi_n(u))du < \varepsilon.$$

Hence tightness, by Lemma 3.1.

(2) Identification of the limit: Assume $Z_{n'} \Rightarrow \widetilde{Z}$, then

$$\mathbf{E}\left(\exp(iu\widetilde{Z})\right) = \lim_{n'\to\infty}\mathbf{E}\left(\exp(iuZ_{n'})\right) = \varphi(u).$$

□

3.3 Erdős–Kac theorem: CLT for number of prime divisors

A mixture of the method of characteristic functions and method of moments.

Denote by $\mathbb{P}$ the set of primes and

$$g : \mathbb{N} \to \mathbb{N}, \qquad g(m) := \#\{p \in \mathbb{P} : p \,|\, m\}.$$

Theorem 3.4 (Paul Erdős & Marc Kac, 1940).

$$\lim_{n\to\infty} n^{-1} \#\Big\{m \in \{1,2,\dots,n\} : \frac{g(m) - \log\log n}{\sqrt{\log\log n}} < x\Big\} = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

Probabilistic setup: Let ω_n be randomly sampled from $(\{1,2,\dots,n\}, UNI)$ and $Z_n := g(\omega_n)$. Then

$$\frac{Z_n - \log\log n}{\sqrt{\log\log n}} \Rightarrow N(0,1).$$

Proof. We will use

$$\sum_{p\in\mathbb{P}:p\le n} \frac{1}{p} = \log\log n + \mathcal{O}(1).$$

Define the random variables $Y_{n,p}$, $p \in \mathbb{P}$, $n \in \mathbb{N}$.

$$Y_{n,p} := \mathbb{1}_{p\,|\,\omega_n}, \quad \text{where} \;\; \omega_n \sim UNI(\{1,2,\dots,n\}).$$

Mind that for $n \in \mathbb{N}$ fixed $\big(Y_{n,p}\big)_{p\in\mathbb{P}}$ are jointly defined.

Then

$$Z_n = \sum_{p\in\mathbb{P}} Y_{n,p}.$$

Note that for any $k < \infty$ and $p_1, p_2, \dots, p_k \in \mathbb{P}$ fixed

$$\big(Y_{n,p_1}, Y_{n,p_2}, \dots, Y_{n,p_k}\big) \Rightarrow \big(X_{p_1}, X_{p_2}, \dots, X_{p_k}\big)$$

where X_p, $p \in \mathbb{P}$, are (jointly defined) *independent* random variables with distribution

$$\mathbf{P}\left(X_p = 1\right) = \frac{1}{p} = 1 - \mathbf{P}\left(X_p = 0\right).$$

How to guess the result? Let

$$\alpha_n \to \infty, \qquad S_n := \sum_{p\in\mathbb{P}:p\le\alpha_n} X_p.$$

Then

$$S_n^* := \frac{S_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}} \Rightarrow N(0,1).$$

Note that

$$\frac{S_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}} = \frac{S_n - \mathbf{E}\left(S_n\right)}{\sqrt{\log\log\alpha_n}} + \frac{\mathbf{E}\left(S_n\right) - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}}$$

and

$$\frac{\mathbf{E}\left(S_n\right) - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}} = \frac{\log\log\log\alpha_n + \mathcal{O}(1)}{\sqrt{\log\log\alpha_n}} \to 0$$

The weak convergence

$$\frac{S_n - \mathbf{E}\left(S_n\right)}{\sqrt{\log\log\alpha_n}} \Rightarrow N(0,1)$$

is proved with method of characteristic functions:

$$\begin{aligned}\mathbf{E}\left(\exp(iuS_n^*)\right) &= \prod_{p\in\mathbb{P}:p\le\alpha_n}\left(\frac{1}{p}\exp\{\frac{iu(p-1)/p}{\sqrt{\log\log\alpha_n}}\} + \frac{p-1}{p}\exp\{\frac{-iu/p}{\sqrt{\log\log\alpha_n}}\}\right)\\ &\to \exp\{-u^2/2\} \qquad\qquad \text{HW!}\end{aligned}$$

Let

$$\begin{aligned}\alpha_n &:= n^{1/\log\log n}\\ \log\alpha_n &= \frac{\log n}{\log\log n}\\ \log\log\alpha_n &= \log\log n - \log\log\log n.\end{aligned}$$

Note that

$$\begin{aligned}&(1): \qquad (\forall\varepsilon > 0) : \alpha_n = o(n^\varepsilon),\\ &(2): \qquad \sum_{\alpha_n<p\le n}\frac{1}{p} = \log\log\log n + \mathcal{O}(1).\end{aligned}$$

Let

$$S_n := \sum_{p\in\mathbb{P}:p\le\alpha_n} X_p, \qquad S_n^* := \frac{S_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}}$$

$$T_n := \sum_{p\in\mathbb{P}:p\le\alpha_n} Y_{n,p}, \qquad T_n^* := \frac{T_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}}$$

$$Z_n := \sum_{p\in\mathbb{P}:p\le n} Y_{n,p} = \sum_{p\in\mathbb{P}} Y_{n,p}, \qquad Z_n^* := \frac{Z_n - \log\log n}{\sqrt{\log\log n}}$$

We know that $S_n^* \Rightarrow N(0,1)$ and we want to prove $Z_n^* \Rightarrow N(0,1)$.
Step 1.

$$\begin{aligned}
\mathbf{E}\left(|Z_n - T_n|\right) &= \sum_{p\in\mathbb{P}:\alpha_n<p\le n} \mathbf{E}\left(Y_{n,p}\right) \le \sum_{p\in\mathbb{P}:\alpha_n<p\le n} \frac{1}{p} \\
&= \log\log\log n + \mathcal{O}(1) = o(\sqrt{\log\log n}) \\
|\log\log n - \log\log\alpha_n| &= \log\log\log n + \mathcal{O}(1) = o(\sqrt{\log\log n})
\end{aligned}$$

Hence

$$|T_n^* - Z_n^*| \xrightarrow{\mathbf{P}} 0.$$

Step 2. We prove $T_n^* \Rightarrow N(0,1)$ with method of moments.
By computation:

$$\lim_{n\to\infty} \mathbf{E}\left(S_n^k\right) = \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} y^k dy =: M_k. \qquad \text{HW!}$$

For $1 < p_1 < p_2 < \dots < p_l \le \alpha_n$ and $k_1, k_2, \dots, k_l \ge 1$:

$$\begin{aligned}
\mathbf{E}\left(X_{p_1}^{k_1} X_{p_2}^{k_2} \dots X_{p_l}^{k_l}\right) &= \mathbf{E}\left(X_{p_1} X_{p_2} \dots X_{p_l}\right) = \frac{1}{p_1 p_2 \dots p_l} \\
\mathbf{E}\left(Y_{n,p_1}^{k_1} Y_{n,p_2}^{k_2} \dots Y_{n,p_l}^{k_l}\right) &= \mathbf{E}\left(Y_{n,p_1} Y_{n,p_2} \dots Y_{n,p_l}\right) = \frac{1}{n}\left\lfloor \frac{n}{p_1 p_2 \dots p_l} \right\rfloor.
\end{aligned}$$

Hence:

$$\left|\mathbf{E}\left(X_{p_1}^{k_1} X_{p_2}^{k_2} \dots X_{p_l}^{k_l}\right) - \mathbf{E}\left(Y_{n,p_1}^{k_1} Y_{n,p_2}^{k_2} \dots Y_{n,p_l}^{k_l}\right)\right| \le \frac{1}{n}.$$

Using this and

$$(x_1 + x_2 + \cdots + x_N)^k =$$
$$= \sum_{l=1}^{N} \sum_{\substack{k_1,k_2,\dots,k_l \geq 1 \\ k_1+k_2+\cdots+k_l=k}} \sum_{1 \leq m_1 < m_2 < \cdots < m_l \leq N} C(l; k_1, k_2, \dots, k_l) x_{m_1}^{k_1} x_{m_2}^{k_2} \dots x_{m_l}^{k_l}$$

we readily obtain

$$\left| \mathbf{E}\left(S_n^k\right) - \mathbf{E}\left(T_n^k\right) \right| \leq \frac{\alpha_n^k}{n} = o(1)$$

and thus

$$\lim_{n\to\infty} \mathbf{E}\left(T_n^k\right) = M_k.$$

Hence:

$$T_n^* \Rightarrow N(0,1),$$

which together with "Step 1" implies

$$Z_n^* \Rightarrow N(0,1),$$

□

3.4 Limit theorem for the coupon collector

Mixture of "bare hands" and characteristic/generating function method.

For $n \in \mathbb{N}$, let $\xi_{n,k}$, $k = 0, 1, \dots, n-1$ be independent geometrically distributed random variables with distribution

$$\mathbf{P}\left(\xi_{n,k} = m\right) = \left(\frac{k}{n}\right)^m \frac{n-k}{n}, \qquad m = 0, 1, 2, \dots$$

and

$$V_n := \sum_{k=0}^{n-1} \xi_{n,k}$$

Then

$$\mathbf{E}\left(\xi_{n,k}\right) = \frac{k}{n-k}, \qquad \mathbf{Var}\left(\xi_{n,k}\right) = \frac{nk}{(n-k)^2}$$
$$\mathbf{E}\left(V_n\right) = n\log n + \mathcal{O}(n), \qquad \mathbf{Var}\left(V_n\right) = \frac{\pi^2}{6}n^2 + \mathcal{O}(n\log n).$$

Theorem 3.5.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{V_n - n\log n}{n} < x\right) = \exp\{-e^{-x}\}.$$

Remark 3.3. *The (two-parameter family of) distributions*

$$\begin{aligned} F_{a,b}(x) &:= \exp\{-e^{-ax+b}\}, \qquad a\in\mathbb{R}_+, b\in\mathbb{R}, \\ f_{a,b}(x) &:= \frac{d}{dx}F_{a,b}(x) = a\exp\{-e^{-ax+b} - ax + b\} \end{aligned}$$

are called Type-1 Gumbel distributions and appear in extreme value theory.

Proof. Let $\zeta_{n,k} := \xi_{n,n-k}$, $k = 1,\dots,n$, and

$$Z_n := \sum_{k=1}^{n}\left(\frac{\zeta_{n,k}}{n} - \frac{1}{k}\right) = \frac{V_n - n\log n}{n} - \gamma + \mathcal{O}(n^{-1}).$$

where γ is Euler's constant

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-1} - \log n\right) \approx 0.5772\dots.$$

Lemma 3.2. *Let $p_n \searrow 0$ so that $np_n \to \lambda \in \mathbb{R}_+$ and ζ_n be a sequence of geometrically distributed random variables with distribution*

$$\mathbf{P}(\zeta_n = r) = (1-p_n)^r p_n.$$

Then $\zeta_n/n \Rightarrow EXP(\lambda)$.

Proof. Straightforward elementary computation. □

Thus

$$\left(\frac{\zeta_{n,1}}{n}, \frac{\zeta_{n,2}}{n}, \dots\right) \Rightarrow (\zeta_1, \zeta_2, \dots)$$

where ζ_k, $k = 1, 2, \dots$ are independent $EXP(k)$-distributed,

$$\mathbf{E}(\zeta_k) = \frac{1}{k}, \qquad \mathbf{Var}(\zeta_k) = \frac{1}{k^2}, \qquad \widetilde{\zeta}_k := \zeta_k - \mathbf{E}(\zeta_k).$$

It follows that

$$Z_n \Rightarrow Z := \lim_{K\to\infty} \sum_{k=1}^{K} \widetilde{\zeta}_k$$

Note that the limit defining Z exists a.s. due to Kolmogorov's inequality (see Probability II.)

Computing the distribution of Z: Let $\Phi : (-1, \infty) \to \mathbb{R}_+$ be the moment generating function (Laplace transform) of Z:

$$\begin{aligned} \Phi(u) &:= \mathbf{E}\left(\exp(-uZ)\right) = \prod_{k=1}^{\infty} \mathbf{E}\left(\exp(-u\widetilde{\zeta}_k)\right) = \cdots \\ &= \exp \sum_{k=1}^{\infty} \big(\log \frac{k}{k+u} + \frac{u}{k} \big) \end{aligned}$$

(Mind that the sum is absolutely convergent!)

Analiticity of $(-1, \infty) \ni u \mapsto \Phi(u)$ and the identities

$$\Phi(0) = 1, \qquad \Phi(u+1) = e^{\gamma}(u+1)\Phi(u) \qquad HW!$$

determine

$$\Phi(u) = e^{\gamma u}\Gamma(u+1).$$

On the other hand:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-uy} d \exp\{-e^{-(y+\gamma)}\} &= \int_{-\infty}^{\infty} e^{-uy} \exp\{-e^{-(y+\gamma)}\} e^{-(y+\gamma)} dy \\ &= e^{\gamma u} \int_0^{\infty} z^u e^{-z} dz \\ &= e^{\gamma u}\Gamma(u+1). \end{aligned}$$

□

Chapter 4

Lindeberg's theorem and its applications

4.1 Triangular arrays of random variables

Let $N_n \in \mathbb{N}$, $\lim_{n\to\infty} N_n = \infty$ and

$$\xi_{n,k}, \quad k = 1, 2, \ldots, N_n, \quad n = 1, 2, \ldots$$

random variables. Explicitly:

$$\begin{array}{llllllll}
\xi_{1,1}, & \ldots, & \xi_{1,N_1} & & & & & \\
\xi_{2,1}, & \xi_{2,2}, & \ldots, & \xi_{2,N_2} & & & & \\
\xi_{3,1}, & \xi_{3,2}, & \xi_{3,3}, & \ldots, & \xi_{3,N_3} & & & \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots & \ldots & & \\
\xi_{n,1}, & \xi_{n,2}, & \xi_{n,3}, & \xi_{n,4}, & \xi_{n,5}, & \ldots, & \xi_{n,N_n} & \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots & \ldots & \ldots & \ldots
\end{array}$$

which are row-wise independent. (Different rows are not even jointly defined.)

Assume:

$$\mathbf{E}\left(\xi_{n,k}\right) = 0, \qquad \mathbf{Var}\left(\xi_{n,k}\right) =: \sigma_{n,k}^2 < \infty$$

and denote their characteristic functions

$$\varphi_{n,k}(u) := \mathbf{E}\left(\exp\{iu\xi_{n,k}\}\right).$$

Let

$$S_n := \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,N_n}.$$

Then

$$\mathbf{E}(S_n) = 0,$$
$$\mathbf{Var}(S_n) = \sigma_{n,1}^2 + \sigma_{n,2}^2 + \cdots + \sigma_{n,N_n}^2 =: \sigma_n^2$$

Question: CLT for $\frac{S_n}{\sigma_n}$?

Theorem 4.1 (Lindeberg, 1922). *If* $(\forall \varepsilon > 0)$

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{N_n} \mathbf{E}\left(|\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n}\right) = 0 \qquad (***)$$

then

$$\frac{S_n}{\sigma_n} \Rightarrow N(0,1).$$

Remarks

Remark 4.1. *Condition (***) is Lindeberg's condition.*

Remark 4.2.

W.l.o.g., we may assume $(\forall n) : \sigma_n = 1$.

Remark 4.3.

The "meaning" of Lindeberg's condition:

"All components $\xi_{n,k}$ *are negligibly tiny compared with* S_n*."*

In particular, it follows that

$$\lim_{n\to\infty} \max_{1\le k\le N_n} \frac{\sigma_{n,k}^2}{\sigma_n^2} = 0. \qquad (*)$$

Indeed,

$$\begin{aligned}
\sigma_{n,k}^2 &= \mathbf{E}\left(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|\le\varepsilon\sigma_n}\right) + \mathbf{E}\left(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n}\right) \\
&\le \varepsilon^2\sigma_n^2 + \mathbf{E}\left(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n}\right).
\end{aligned}$$

BUT: condition (*) is genuinely weaker then (***) and it is not sufficient for the CLT to hold!

Remark 4.4.

The old CLT for sums of i.i.d. random variables ζ_k *follows with* $\xi_{n,k} := \zeta_k$.

Remark 4.5.

In a very precise sense: Condition (***) *is sufficient and necessary for the CLT to hold (W. Feller).*

Proof. W.l.o.g., we assume $(\forall n): \quad \sigma_n = 1$ and plan to prove:

$$(\forall u): \qquad \lim_{n\to\infty} \prod_{k=1}^{N_n} \varphi_{n,k}(u) = e^{-u^2/2}.$$

Lemma 4.1.

$$(\forall t \in \mathbb{R}): \qquad \left|e^{it} - \sum_{l=0}^{m} \frac{(it)^l}{l!}\right| \le \min\left\{\frac{|t|^{m+1}}{(m+1)!}, \frac{2|t|^m}{m!}\right\}.$$

Proof of Lemma 4.1. By induction on m:

$$\begin{aligned} e^{it} - \sum_{l=0}^{m} \frac{(it)^l}{l!} &= \frac{i^{m+1}}{m!} \int_0^t (t-s)^m e^{is} ds \\ &= \frac{i^m}{(m-1)!} \int_0^t (t-s)^{m-1}(e^{is}-1) ds \end{aligned}$$

□

It follows that

$$\begin{aligned} \left|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2}\right| &= \left|\mathbf{E}\left(e^{iu\xi_{n,k}} - \sum_{l=0}^{2} \frac{(iu\xi_{n,k})^l}{l!}\right)\right| \\ &\le \mathbf{E}\left(\left|e^{iu\xi_{n,k}} - \sum_{l=0}^{2} \frac{(iu\xi_{n,k})^l}{l!}\right|\right) \\ &\le \mathbf{E}\left(\min\{|u\xi_{n,k}|^3/6, |u\xi_{n,k}|^2\}\right) \\ &\le \frac{|u|^3}{6}\mathbf{E}\left(|\xi_{n,k}|^3 \mathbb{1}_{|\xi_{n,k}|\le\varepsilon}\right) + \\ &\qquad + |u|^2 \mathbf{E}\left(|\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon}\right) \\ &\le \frac{\varepsilon|u|^3}{6}\sigma_{n,k}^2 + |u|^2 \mathbf{E}\left(|\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon}\right). \end{aligned}$$

Hence, using (***),

$$\lim_{n\to\infty}\sum_{k=1}^{N_n}\left|\varphi_{n,k}(u)-1+\frac{u^2\sigma_{n,k}^2}{2}\right|=0. \tag{4.1}$$

This is the main point of the proof!

$$\begin{aligned}
\left|\sum_{k=1}^{N_n}\log\varphi_{n,k}(u)+\frac{u^2}{2}\right| &= \left|\sum_{k=1}^{N_n}\Big(\log\varphi_{n,k}(u)+\frac{u^2\sigma_{n,k}^2}{2}\Big)\right| \\
&\leq \sum_{k=1}^{N_n}\left|\log\varphi_{n,k}(u)+\frac{u^2\sigma_{n,k}^2}{2}\right| \\
&\leq \sum_{k=1}^{N_n}\left|\log\varphi_{n,k}(u)+(1-\varphi_{n,k}(u))\right|+ \\
&\qquad +\sum_{k=1}^{N_n}\left|\varphi_{n,k}(u)-1+\frac{u^2\sigma_{n,k}^2}{2}\right| \qquad (4.2)
\end{aligned}$$

We show that the last two sums go to zero, as $n\to\infty$.

From (4.1) it follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\max_{1\leq k\leq N_n}\left|\varphi_{n,k}(u)-1\right| \\
&\leq \limsup_{n\to\infty}\max_{1\leq k\leq N_n}\left|\varphi_{n,k}(u)-1+\frac{u^2\sigma_{n,k}^2}{2}\right| \\
&\quad+\limsup_{n\to\infty}\max_{1\leq k\leq N_n}\frac{u^2\sigma_{n,k}^2}{2} \\
&=0.
\end{aligned} \tag{4.3}$$

This implies that for $n\geq n_0$ and $1\leq k\leq N_n$:

$$|\varphi_{n,k}(u)-1|<1/2,$$

and

$$\begin{aligned}
&\left|\log\varphi_{n,k}(u)+1-\varphi_{n,k}(u)\right|\leq|1-\varphi_{n,k}(u)|^2 \\
&\quad\leq\left(\max_{1\leq k'\leq N_n}|1-\varphi_{n,k'}(u)|\right)\left(\left|\varphi_{n,k}(u)-1+\frac{u^2\sigma_{n,k}^2}{2}\right|+\frac{u^2\sigma_{n,k}^2}{2}\right).
\end{aligned}$$

Hence

$$\sum_{k=1}^{N_n} \big| \log \varphi_{n,k}(u) + (1 - \varphi_{n,k}(u)) \big|$$
$$\leq \left(\max_{1\leq k'\leq N_n} |1 - \varphi_{n,k'}(u)| \right) \left(\sum_{k=1}^{N_n} \Big|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2} \Big| + \frac{u^2}{2} \right).$$

Now, from (4.1), (4.2) and (4.3) it follows that

$$\lim_{n\to\infty} \Big| \sum_{k=1}^{N_n} \log \varphi_{n,k}(u) + \frac{u^2}{2} \Big| = 0.$$

□

4.2 Application 1: CLT for the number of records

Let η_k, $k = 1, 2, \dots$ be i.i.d., $\eta_k > 0$, with continuous distrib., and

$$\xi_1 = 1, \quad \xi_k := 1\!\!1_{\eta_k > \max_{1\leq j<k} \eta_j}, \quad k > 1, \quad S_n := \xi_1 + \cdots + \xi_n.$$

Then ξ_k, $k = 1, 2, \dots$ are *independent* (HW!) with distribution

$$\mathbf{P}(\xi_k = 1) = \frac{1}{k} = 1 - \mathbf{P}(\xi_k = 0),$$

$$\mathbf{E}(\xi_k) = \frac{1}{k}, \qquad \mathbf{Var}(\xi_k) = \frac{k-1}{k^2},$$
$$\mathbf{E}(S_n) = \log n + \mathcal{O}(1), \qquad \mathbf{Var}(S_n) = \log n + \mathcal{O}(1).$$

Theorem 4.2.
$$\frac{S_n - \log n}{\sqrt{\log n}} \Rightarrow N(0,1).$$

4.3 Application 2: CLT in the "borderline" case

Let η_k, $k = 1, 2 \dots$ i.i.d. with distribution density

$$\frac{d}{dx}\mathbf{P}(\eta_j < x) =: f(x) = |x|^{-3}\mathbb{1}_{|x|>1}.$$

Then

$$(\forall \varepsilon > 0): \ \mathbf{E}\left(|\eta_j|^{2-\varepsilon}\right) < \infty, \qquad \mathbf{E}(\eta_j) = 0, \qquad \mathbf{E}\left(|\eta_j|^2\right) = \infty.$$

Theorem 4.3.
$$\frac{\eta_1 + \dots + \eta_n}{\sqrt{n \log n}} \Rightarrow N(0, 1).$$

Proof. Define

$$\xi_{n,k} := \eta_k \mathbb{1}_{|\eta_k| < \sqrt{n} \log\log n}, \qquad k = 1, 2, \dots, n$$

And apply Lindeberg's Theorem for the triangular array $\xi_{n,k}$, $k = 1, 2, \dots n$, $n = 1, 2, \dots$.
Mind that $(\forall n) : \xi_{n,k}$, $k = 1, 2, \dots n$ are i.i.d.

$$\begin{aligned}
&\mathbf{E}(\xi_{n,k}) = 0, \\
&\sigma_{n,k}^2 = \sigma_{n,1}^2 = \log n + 2\log\log\log n, \\
&\sigma_n^2 = n\sigma_{n,1}^2.
\end{aligned}$$

Lindeberg's condition (***):

$$\lim_{n\to\infty} \frac{1}{\sigma_{n,1}^2} \mathbf{E}\left(|\xi_{n,1}|^2 \mathbb{1}_{|\xi_{n,1}|^2 > \varepsilon n \sigma_{n,1}^2}\right) = 0$$

holds because

$$|\xi_{n,1}|^2 < n(\log\log n)^2 < \varepsilon n \sigma_{n,1}^2.$$

So:

$$\frac{\xi_{n,1} + \cdots + \xi_{n,n}}{\sqrt{n \log n}} \Rightarrow N(0,1). \tag{4.4}$$

What is the error made with the cutoff?

$$\mathbf{P}\left((\exists k \leq n) : \xi_{n,k} \neq \eta_k\right) \leq n\mathbf{P}\left(\xi_{n,1} \neq \eta_1\right) = \frac{1}{(\log\log n)^2} \to 0$$

Hence

$$\sum_{k=1}^{n} |\eta_k - \xi_{n,k}| \xrightarrow{\mathbf{P}} 0. \tag{4.5}$$

The theorem follows form (4.4) and (4.5). □

Chapter 5

Stable distributions and stable limits

5.1 Affine equivalence

Definition 5.1. *The probability distributions* $F_1, F_2 : \mathbb{R} \to [0,1]$ *are affine-equivalent iff*

$$\big(\exists\ a \in (0,\infty), b \in \mathbb{R}\big) : \quad \big(\forall\ x \in \mathbb{R}\big) : \quad F_2(x) = F_1(ax+b).$$

Remarks

Remark 5.1. *This is clearly an equivalence relation. A class of equivalence can be parametrized as*

$$(0,\infty) \times (-\infty,\infty) \ni (a,b) \mapsto F_{a,b}(\cdot) := F_{1,0}(a\cdot+b).$$

Remark 5.2. *In terms of the random variables* X_1, X_2 *(of disrib.* F_1, F_2*):*

$$\big(\exists\ a \in (0,\infty), b \in \mathbb{R}\big) : \quad X_2 \sim aX_1 + b.$$

Remark 5.3. *In terms of the characteristic functions φ_1, φ_2 (of the distributions F_1, F_2):*

$$\big(\exists\ a \in (0,\infty), b \in \mathbb{R}\big) : \quad \big(\forall\ u \in \mathbb{R}\big) : \quad \varphi_2(u) = e^{ibu}\varphi_1(au).$$

5.2 Stability

Definition 5.2. *An affine-equivalent class of distributions is stable iff it is closed under convolution. A distribution is called stable if it belongs to a stable class: the distribution F is stable iff*

$$\big(\forall\, a_1, a_2 > 0\big) : \quad \big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\big) : \quad F(a_1\cdot) * F(a_2\cdot) = F(a_3 \cdot + b_3).$$

Remarks

Remark 5.4. *In terms of the random variables:*

$$\big(\forall\, a_1, a_2 > 0\big) : \quad \big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\big) : \quad a_1X_1 + a_2X_2 = a_3X_3 + b_3,$$

where $X_1, X_2, X_3 \sim F$ and X_1, X_2 are independent.

Remark 5.5. *In terms of the characteristic function:*

$$\big(\forall\, a_1, a_2 > 0\big) : \quad \big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\big) : \quad \varphi(a_1u)\varphi(a_2u) = e^{ib_3u}\varphi(a_3u).$$

Remark 5.6. *By induction it follows that*

$$\big(\forall\ k \in \mathbb{N},\ a_1, \dots, a_k > 0\big) : \quad \big(\exists\ a_{k+1} > 0,\ b_{k+1} \in \mathbb{R}\big) :$$
$$a_1X_1 + \cdots + a_kX_k = a_{k+1}X_{k+1} + b_{k+1},$$

where $X_1, \dots, X_k, X_{k+1} \sim F$ and $X_1, \dots, X_k$ are independent.

5.3 Examples

Example 5.1 (Counterexample). *Discrete distributions CAN'T BE stable. Actually: a stable distribution doesn't have point mass. (Obvious!)*

Example 5.2. *The class of Gaussian (normal) distributions is stable:*

$$\sigma > 0, m \in \mathbb{R}: \quad f_{\sigma,m}(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\{-(x-m)^2/(2\sigma^2)\},$$
$$\varphi_{\sigma,m}(u) = \exp\{imu - \frac{\sigma^2 u^2}{2}\}.$$

Indeed, for $\sigma_1, \sigma_2 > 0$ *and* $m_1, m_2 \in \mathbb{R}$

$$f_{\sigma_1,m_1} * f_{\sigma_2,m_2} = f_{\sigma_3,m_3}, \qquad \varphi_{\sigma_1,m_1}\varphi_{\sigma_2,m_2} = \varphi_{\sigma_3,m_3},$$

with

$$\sigma_3 = (\sigma_1^2 + \sigma_2^2)^{1/2}, \qquad m_3 = m_1 + m_2.$$

Example 5.3. *The class of Cauchy distributions is stable:*

$$\tau > 0, m \in \mathbb{R}: \quad f_{\tau,m}(x) := \frac{1}{\pi\tau} \cdot \frac{1}{1 + (x-m)^2/\tau^2},$$
$$\varphi_{\tau,m}(u) = \exp\{imu - \tau|u|\}.$$

Indeed, for $\tau_1, \tau_2 > 0$ *and* $m_1, m_2 \in \mathbb{R}$

$$f_{\tau_1,m_1} * f_{\tau_2,m_2} = f_{\tau_3,m_3}, \qquad \varphi_{\tau_1,m_1}\varphi_{\tau_2,m_2} = \varphi_{\tau_3,m_3},$$

with

$$\tau_3 = \tau_1 + \tau_2, \qquad m_3 = m_1 + m_2.$$

Example 5.4. *Recall the distribution of first hitting times of Brownian motion: B_t standard $1d$ Brownian motion, $B_0 = 0$.*

$$T_r := \inf\{t : B_t = r\} \overset{(1)}{\sim} r^2 \inf\{t : B_t = 1\}$$
$$f_{\alpha,0}(s) := \frac{\partial}{\partial s}\mathbf{P}\left(T_{\sqrt{\alpha}} < s\right) = \frac{\partial}{\partial s}\mathbf{P}\left(T_1 < s/\alpha\right).$$

(1) : By scaling of Brownian motion.

Then,

$$\alpha > 0, m = 0 : \quad f_{\alpha,0}(s) \overset{(2)}{=} \frac{1}{\sqrt{2\pi}\alpha}(s/\alpha)^{-3/2}e^{-\alpha/(2s)}\mathbb{1}_{s>0},$$
$$\varphi_{\alpha,0}(u) \overset{(3)}{=} \exp\{-(1+i)\alpha^{1/2}|u|^{1/2}\}.$$

(2) : See earlier work. (Max. and hitting times of RW and BM.)

(3) : Computation. Will be done later. Try it as HW.

Indeed, for $\alpha_1, \alpha_2 > 0$

$$f_{\alpha_1,0} * f_{\alpha_2,0} \overset{(4)}{=} f_{\alpha_3,0}, \qquad \varphi_{\alpha_1,0}\varphi_{\alpha_2,0} = \varphi_{\alpha_3,0},$$

with

$$\alpha_3 = (\alpha_1^{1/2} + \alpha_2^{1/2})^2.$$

(4) : By independent + stationary increments and scaling of Brownian motion:

$$(r_1 + r_2)^2 T_1 \ \sim \ T_{r_1+r_2} \ \sim \ T_{r_1} + T'_{r_2} \ \sim \ r_1^2 T_1 + r_2^2 T'_1$$

where T_{r_1} and T'_{r_2}, respectively, T_1 and T'_1 are independent.

Homework 5.1. *Let (X_t, Y_t) be standard $2d$ Brownian motion, starting form $(X_0, Y_0) = (0, 0)$, and $T_1 := \inf\{t : X_t = 1\}$. Compute the distribution of Y_{T_1}.*

Proposition 5.1. *The distribution* F *is stable if and only if for any* $k \in \mathbb{N}$ *there exist* $\alpha_k > 0, \beta_k \in \mathbb{R}$ *such that*

$$X_1 + \cdots + X_k = \alpha_k X + \beta_k,$$

where $X_1, \dots, X_k, X \sim F$ *and* $X_1, \dots, X_k$ *are independent.*

Proof of the Proposition: Later. □

Limit laws of centred and normed sums of i.i.d. random variables are always stable:

Theorem 5.1. *Let* $X_1, X_2, \dots$ *be i.i.d. random variables and* $S_n := X_1 + \cdots + X_n$*. If there exist (deterministic) sequences* $a_n > 0$ *and* $b_n \in \mathbb{R}$ *such that*

$$Z_n := \frac{S_n - b_n}{a_n} \Rightarrow Y,$$

as $n \to \infty$*, then the distribution of* Y *is stable.*

Proof. Fix $k \in \mathbb{N}$ and denote,

$$\sum_{i=n(j-1)+1}^{nj} X_i =: S_n^{(j)} \;\sim\; S_n. \qquad j = 1, \dots, k.$$

Then

$$S_{kn} = S_n^{(1)} + \cdots + S_n^{(k)}$$

and

$$\frac{a_{kn}}{a_n} Z_{kn} - \frac{kb_n - b_{kn}}{a_n} = Z_n^{(1)} + \cdots + Z_n^{(k)},$$

where $Z_n^{(1)}, \cdots, Z_n^{(k)} \;\sim\; Z_n$ are i.i.d. Thus:

$$Z_{kn} \Rightarrow Y, \qquad Z_n^{(1)} + \cdots + Z_n^{(k)} \Rightarrow Y^{(1)} + \cdots + Y^{(k)},$$

as $n \to \infty$, where $Y^{(1)}, \dots, Y^{(k)} \;\sim\; Y$ are i.i.d.

Lemma 5.1. *Let* W_n *be a sequence of random variables,* $\alpha_n > 0$*,* $\beta_n \in \mathbb{R}$ *(deterministic) sequences, and* $W_n' := \alpha_n W_n + \beta_n$*. If both* $W_n \Rightarrow W$ *and* $W_n' \Rightarrow W'$*, as* $n \to \infty$ *where* W *and* W' *are both nondegenerate random variables, then the limits* $\lim_{n\to\infty} \alpha_n =: \alpha > 0$ *and* $\lim_{n\to\infty} \beta_n =: \beta \in \mathbb{R}$ *exist.*

Proof of Lemma 5.1. Easy: write the characteristic functions. □

By Lemma 5.1, the limits

$$\lim_{n\to\infty} \frac{a_{kn}}{a_n} =: \alpha_k > 0, \qquad \lim_{n\to\infty} \frac{kb_n - b_{kn}}{a_n} =: \beta_k \in \mathbb{R}$$

exist, and thus

$$Y^{(1)} + \cdots + Y^{(k)} \sim \alpha_k Y - \beta_k.$$

Theorem 5.1 follows from Proposition 5.1. □

5.4 Symmetric stable laws

Easier then the general case.

Theorem 5.2. *(i) Let $c > 0$ and $\alpha \in (0, 2]$. The function $\varphi : \mathbb{R} \to \mathbb{R}$*

$$\varphi(u) := \exp\{-c|u|^\alpha\} \tag{5.1}$$

is characteristic function of a symmetric stable distribution.

(ii) The characteristic function of a symmetric stable distribution is of the form (5.1), with some $c > 0$ and $\alpha \in (0, 2]$.

Remarks

Remark 5.7. *The parameter $c > 0$ can be changed by scaling. The parameter $\alpha \in (0, 2]$ is* essential. *It is called the index of the stable law.*

Remark 5.8. *$u \mapsto \varphi(u)$ of (5.1) obviously satisfies the stability condition. It is to be checked that*

- *It is indeed a characteristic function.*
- *There are no other chf-s of symmetric stable laws.*

Remark 5.9. *Examples: $\alpha = 2$: Gaussian; $\alpha = 1$: Cauchy. No explicit formula for the distribution function/density in other cases.*

Remark 5.10. *In the symmetric stable case:*

$$a_1X_1 + a_2X_2 \;\sim\; \left(a_1^\alpha + a_2^\alpha\right)^{1/\alpha} X.$$

Proof of (i) in Theorem 5.2, for $\alpha \in (0,1]$.

Theorem 5.3 (György Pólya's construction). *Let $\varphi : \mathbb{R} \to [0,1]$ satisfy:*

- $\lim_{u\to 0} \varphi(u) = 1$, • $\varphi(-u) = \varphi(u)$, • $[0,\infty) \ni u \mapsto \varphi(u)$ *convex.*

Then φ is a characteristic function.

If $\alpha \in (0,1]$, then $\varphi(u)$ of (5.1) is of this form. □

Proof of Pólya's theorem.

$$\psi_1(u) := \left(1 - |u|\right)_+ = \int_{-\infty}^{\infty} e^{iux}\, \frac{1-\cos x}{\pi x^2} dx,$$
$$\psi_a(u) := \psi_1(au) = \left(1 - a|u|\right)_+, \qquad a > 0,$$

are characteristic functions. The functions of the theorem are *pointwise limits* of functions of the form

$$u \mapsto \sum_{k=1}^{K} p_k \psi_{a_k}(u),$$

with

$$a_1, \dots, a_K > 0; \qquad p_1, \dots, p_K \in [0,1], \quad p_1 + \dots + p_K = 1,$$

which are themselves characteristic functions. □

Proof of (i) in Theorem 5.2, for $\alpha \in (0,2)$. Let $X_1, X_2, \dots$ be i.i.d. with symmetric distribution density f:

$$f(x) := \frac{\alpha}{2|x|^{\alpha+1}} 1\!\!1_{|x|>1},$$

and characteristic function ψ. Then

$$1 - \psi(u) = \alpha \int_1^{\infty} \frac{1-\cos(ux)}{x^{\alpha+1}} dx = \alpha|u|^\alpha \int_{|u|}^{\infty} \frac{1-\cos y}{y^{\alpha+1}} dy$$

Since $0 < \alpha < 2$ (!):

$$\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}} dy =: \frac{c}{\alpha} < \infty,$$
$$\int_0^{|u|} \frac{1-\cos y}{y^{\alpha+1}} dy = \mathcal{O}(|u|^{2-\alpha}).$$

Thus,

$$\psi(u) = 1 - c|u|^\alpha + \mathcal{O}(|u|^2),$$

and hence, for any $u \in \mathbb{R}$ fixed

$$\mathbf{E}\left(\exp\{iu\frac{S_n}{n^{1/\alpha}}\}\right) = \psi(un^{-1/\alpha})^n$$
$$= \big(1 - \frac{c|u|^\alpha}{n} + \mathcal{O}(n^{-2/\alpha})\big)^n \to e^{-c|u|^\alpha}.$$

We have proved that $u \mapsto e^{-c|u|^\alpha}$ is the characteristic function of a symmetric stable distribution F and the limit theorem

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{S_n}{n^{1/\alpha}} < x\right) = F(x).$$

□

Proof of (ii) in Theorem 5.2. We prove that if φ is characteristic function of a symmetric stable law then it is of the form (5.1).

Let F be a symmetric stable law and φ its characteristic function.

Lemma 5.2. *[Some basic facts about φ]*

(i)

$$\big(\forall u \in \mathbb{R}\big): \quad \varphi(u) = \overline{\varphi(u)} = \varphi(-u).$$

(ii)

$$\big(\forall u \in \mathbb{R}\big): \quad \varphi(u) > 0.$$

(iii) If $b > a > 0$ then

$$\big(\exists u \in \mathbb{R}\big): \quad \varphi(bu) \neq \varphi(au).$$

Proof of the Lemma 5.2. (i) It follows from symmetry of the distribution F.

(ii) Due to symmetry and stability,

$$\big(\exists\ c \in (0,1) \cup (1,\infty)\big) : \ \big(\forall\ u \in \mathbb{R}\big) : \ \varphi(u)^2 = \varphi(cu)$$

(If $c = 1$ then $(\forall\ u \in \mathbb{R}) : \ \varphi(u)^2 = \varphi(u)$, and, by continuity at $u = 0$, $\varphi(u) \equiv 1$. This case is excluded.)

$$\big\{\varphi(u_0) = 0\big\} \ \Rightarrow\ \big\{\big(\forall\ k \in \mathbb{Z}\big) : \ \varphi(c^k u_0) = 0\big\}.$$

This is impossible, due to continuity at $u = 0$.

(iii) This holds for any characteristic function.
Let $c := a/b < 1$. By continuity at $u = 0$

$$\big\{\big(\forall\ u \in \mathbb{R}\big) : \ \varphi(u) = \varphi(cu)\big\} \ \Rightarrow\ \big\{\varphi(u) \equiv 1\big\}.$$

But this case is excluded.

□

By symmetric stability there exists

$$\gamma : \mathbb{N} \to \mathbb{R}_+, \qquad \big(\forall\ u \in \mathbb{R}\big) : \ \ \varphi(u)^n = \varphi(\gamma(n)u).$$

We get

$$\big(\forall\ u \in \mathbb{R}\big) : \quad \varphi(\gamma(nm)u) = \varphi(u)^{nm} = \varphi(\gamma(n)\gamma(m)u)$$

Hence, by (iii) of Lemma 5.2,

$$\gamma(nm) = \gamma(n)\gamma(m).$$

Extend

$$\gamma : \mathbb{Q} \to \mathbb{R}_+, \qquad \gamma(n/m) := \gamma(n)/\gamma(m).$$

Then

$$\big(\forall\ u \in \mathbb{R}\big) : \quad \varphi(u)^{n/m} = \cdots = \varphi(\gamma(n/m)u)$$

Let $r \in \mathbb{R}_+$ and $r_n \in \mathbb{Q}$, $\lim_{n\to\infty} r_n = r$.
Then

$$\varphi(\gamma(r_n)u) = \varphi(u)^{r_n} \to \varphi(u)^r.$$

$\gamma(r_{n'}) \to 0$ or $\gamma(r_{n'}) \to \infty$ implies $\varphi(u) \equiv 1$ – impossible.

Similarly, if $\gamma(r_{n'}) \to g' \in \mathbb{R}$ and $\gamma(r_{n''}) \to g'' \in \mathbb{R}$ then again by (iii) of the Lemma $g' = g''$. So, we extend $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\big(\forall\ u \in \mathbb{R}\big) : \quad \varphi(u)^r = \varphi(\gamma(r)u) \tag{5.2}$$

$$\gamma(rs) = \gamma(r)\gamma(s). \tag{5.3}$$

$$r \mapsto \gamma(r) \quad \text{is continuous.} \tag{5.4}$$

Lemma 5.3 ("Cauchy's problem"). *Let $\gamma : (0,\infty) \to (0,\infty)$ satisfy (5.3) and (5.4). Then $\gamma(r) = r^\beta$ for some $\beta \in \mathbb{R}$.*

From (5.2) it follows that $\varphi(u) = \exp\{-c|u|^\alpha\}$ with $\alpha = 1/\beta$.

$c \leq 0$ or $\alpha \notin (0,2]$ are a priori excluded. □

Remarks

Remark 5.11. *The symmetric stable distributions are absolutely continuous with C^∞ density functions.*

Remark 5.12. *"Heavy tail": For $\alpha \in (0,2)$:*

$$F'(x) =: f(x) \sim C(\alpha)|x|^{-\alpha-1}, \qquad \text{as} \quad |x| \to \infty$$

Remark 5.13. *In particular*

$$(\forall \varepsilon > 0) : \ \mathbf{E}\left(|X|^{\alpha-\varepsilon}\right) < \infty,$$
$$\mathbf{E}\left(|X|^{\alpha}\right) = \infty.$$

Theorem 5.4. *Let $X_1, X_2, \ldots$ be i.i.d. random variables. Denote their (common) distribution function by F and $S_n := X_1 + \cdots + X_n$. Assume that the distribution F is symmetric*

$$F(-x) = 1 - F(x+0),$$

and the tail of the distribution has regular power-law asymptotics

$$\lim_{x\to\infty} x^\alpha \big(1 - F(x)\big) = b,$$

with $\alpha \in (0,2)$ and $b \in (0,\infty)$. Then

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\{iuS_n/n^{1/\alpha}\}\right) = e^{-c|u|^\alpha},$$

with

$$c = 2b\alpha \int_0^\infty \frac{1-\cos y}{y^{\alpha+1}} dy.$$

Remark 5.14. *This Theorem extends the earlier construction. A more general theorem will be stated later.*

Proof. We prove for $|u| \ll 1$

$$\psi(u) := \mathbf{E}\left(\exp\{iuX_j\}\right) = 1 - c|u|^\alpha + o(|u|^\alpha), \qquad (***)$$

and hence

$$\begin{aligned}\mathbf{E}\left(\exp\{iuS_n/n^{1/\alpha}\}\right) &= \left(\psi(u/n^{1/\alpha})\right)^n \\ &= \left(1 - c|u|^\alpha/n + o(1/n)\right)^n \to e^{-c|u|^\alpha}.\end{aligned}$$

Proof of (***) follows:
Fix $\varepsilon > 0$, at the end of the proof we let $\varepsilon \to 0$.

$$\begin{aligned}1 - \psi(u) &= 2\int_0^\infty \big(1-\cos(ux)\big) dF(x) \\ &= 2\int_0^{1/(\varepsilon u)} \big(1-\cos(ux)\big) dF(x) + 2\int_{1/(\varepsilon u)}^\infty \big(1-\cos(ux)\big) dF(x).\end{aligned}$$

Further,

$$\begin{aligned}
2\int_0^{1/(\varepsilon u)} & \big(1-\cos ux\big)dF(x) \\
&= 2\int_0^{1/(\varepsilon u)} \big(1-\cos ux\big)d(F(x)-1) \\
&= 2\big(1-\cos(1/\varepsilon)\big)\big(F(1/(\varepsilon u))-1\big) + 2u\int_0^{1/(\varepsilon u)} \big(1-F(x)\big)\sin(ux)dx.
\end{aligned}$$

Altogether

$$1-\psi(u) = A(u,\varepsilon)+B(u,\varepsilon)+C(u,\varepsilon),$$

where

$$\begin{aligned}
A(u,\varepsilon) &:= 2\int_{1/(\varepsilon u)}^{\infty} \big(1-\cos(ux)\big)dF(x), \\
B(u,\varepsilon) &:= 2\big(1-\cos(1/\varepsilon)\big)\big(F(1/(\varepsilon u))-1\big), \\
C(u,\varepsilon) &:= 2u\int_0^{1/(\varepsilon u)} \big(1-F(x)\big)\sin(ux)dx.
\end{aligned}$$

We keep $\varepsilon>0$ fixed. Then clearly,

$$\max\{|A(u,\varepsilon)|,|B(u,\varepsilon)|\} \le 4\big(1-F(1/(\varepsilon u))\big) = 4b\varepsilon^\alpha|u|^\alpha + o(|u|^\alpha).$$

$$\begin{aligned}
C(u,\varepsilon) &\overset{(1)}{=} 2|u|^\alpha \int_0^{1/\varepsilon} (y/u)^\alpha\big(1-F(y/u)\big)\frac{\sin y}{y^\alpha}dy \\
&\overset{(2)}{=} 2b|u|^\alpha \int_0^{1/\varepsilon} \frac{\sin y}{y^\alpha}dy + o(|u|^\alpha) \\
&\overset{(3)}{=} 2b\alpha|u|^\alpha \int_0^{1/\varepsilon} \frac{1-\cos y}{y^{\alpha+1}}dy + 2b|u|^\alpha\varepsilon^\alpha\big(1-\cos(1/\varepsilon)\big) + o(|u|^\alpha) \\
&\overset{(4)}{=} 2b\alpha|u|^\alpha \int_0^{\infty} \frac{1-\cos y}{y^{\alpha+1}}dy - 2b\alpha|u|^\alpha \int_{1/\varepsilon}^{\infty} \frac{1-\cos y}{y^{\alpha+1}}dy + \\
&\qquad + 2b|u|^\alpha\varepsilon^\alpha\big(1-\cos(1/\varepsilon)\big) + o(|u|^\alpha)
\end{aligned}$$

(1) : Change of variable $y := ux$

(2) : Dominated convergence.

(3) : Integration by parts.

(4) : Absolute integrability.

Altogether, with *any* $\varepsilon > 0$ fixed:

$$\left|1 - \psi(u) - c|u|^\alpha\right| \le o(|u|^\alpha) + 16b\varepsilon^\alpha|u|^\alpha.$$

Hence (***). □

5.5 Examples, applications

Example 5.5 (Sums of reciprocals of absolutely continuous i.i.d. r.v.-s). *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with absolutely continuous distribution. Denote their density function* f *and assume that* f *is continuous at* $x = 0$ *and* $f(0) \in (0, \infty)$. *Then*

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{X_k} \Rightarrow CAU(0, \tau),$$

with some $\tau \in (0, \infty)$.

Example 5.6 (Holtzmark's first (one dimensional) problem). *For* $n \in \mathbb{N}$, *let* $X_{n,1}, \ldots, X_{n,n}$ *be i.i.d.* $UNI[-n/2, n/2]$. *These are "positions of stars or charges". A star/charge located at* $x \in \mathbb{R}$ *generates at the origin the force*

$$F(x) = \text{sgn}(x)|x|^{-p}.$$

So, the resulting total force *generated by the system of* n *randomly positioned stars at the origin is*

$$R_n = \sum_{k=1}^{n} \text{sgn}(X_{n,k})|X_{n,k}|^{-p}.$$

Question: Does R_n *have a limiting distribution, as* $n \to \infty$*?*

Theorem 5.5. *If* $1/2 < p < \infty$ *then*

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\{iuR_n\}\right) = e^{-c|u|^{1/p}}$$

with

$$c = HW!.$$

Proof. Let $Y_1, Y_2, \ldots$ be i.i.d. $UNI[-1/2, 1/2]$-distributed. Then

$$\{X_{n,1}, \ldots, X_{n,n}\} \sim \{nY_1, \ldots, nY_n\}$$

$$R_n \sim n^{-p} \sum_{k=1}^{n} \operatorname{sgn}(Y_k)|Y_k|^{-p}.$$

Note that $\xi_k := \operatorname{sgn}(Y_k)|Y_k|^{-p}$, $k = 1, 2, 3, \ldots$ are i.i.d., symmetric and

$$\mathbf{P}\left(\operatorname{sgn}(Y_k)|Y_k|^{-p} > x\right) = \mathbf{P}\left(0 < Y_k < x^{-1/p}\right) = x^{-1/p}$$

and the limit theorem is applied. □

Example 5.7 (Holtzmark's second (multi-dimensional) problem). *Identical stars/charges are located in* $\mathbb{R}^d$ *according to a homogeneous Poisson point process (PPP) of density* ρ. *Denote their locations* $\vec{X}_1^{(\rho)}, \vec{X}_2^{(\rho)}, \dots$ *in some (arbitrary) ordering. A star/charge located at* $\vec{x} \in \mathbb{R}^d \setminus \vec{0}$ *generates at the origin the force*

$$\vec{F}(\vec{x}) = |\vec{x}|^{-p-1}\vec{x} = \vec{\mathrm{sgn}}(\vec{x})|\vec{x}|^{-p}.$$

Denote by $\vec{R}^{(\rho)}$ *the resulting total force generated at the origin by all stars/charges. Formally:*

$$\vec{R}^{(\rho)} = \sum_i \vec{F}(\vec{X}_i^{(\rho)}).$$

Note that convergence problems may arise.

Question: Assuming that $\vec{R}^{(\rho)}$ *makes some sense, can we say something about its distribution?*

1. *Scaling:*

$$\left(\vec{X}_1^{(\rho)}, \vec{X}_2^{(\rho)}, \dots\right) \sim \left(\rho^{-1/d}\vec{X}_1^{(1)}, \rho^{-1/d}\vec{X}_2^{(1)}, \dots\right)$$
$$(\forall a > 0): \qquad \vec{F}(a\vec{x}) = a^{-p}\vec{F}(\vec{x})$$

It follows that

$$\vec{R}^{(\rho)} \sim \rho^{p/d}\vec{R}^{(1)} \tag{5.5}$$

2. *"Independent increments": If* $PPP^{(\rho_1)}$ *and* $PPP^{(\rho_2)}$ *are two* independent *Poisson point processes of density* ρ_1, *respectively,* ρ_2 *then*

$$PPP^{(\rho_1)} \cup PPP^{(\rho_2)} \sim PPP^{(\rho_1+\rho_2)}. \tag{5.6}$$

From (5.5) and (5.6) it follows that:

$$\rho_1^{p/d}\vec{R}' + \rho_2^{p/d}\vec{R}'' = (\rho_1 + \rho_2)^{p/d}\vec{R}'''$$

where $\vec{R}' \sim \vec{R}'' \sim \vec{R}''' \sim \vec{R}^{(1)}$, $\vec{R}'$ *and* $\vec{R}''$ *are independent, and* $\rho_1, \rho_2 > 0$.

If $\vec{R}^{(\rho)}$ does make sense then it has symmetric stable distribution of index

$$\alpha = \frac{d}{p}, \qquad \frac{d}{2} < p < \infty.$$

Remarks

Remark 5.15. *The summation should be done as*

$$\vec{R} = \lim_{n\to\infty} \sum_i \vec{F}(\vec{X}_i) \mathbb{1}_{\vec{X}_i \in \Lambda_n},$$

where Λ_n is a sequence of increasing, symmetric *domains, $\cup_n \Lambda_n = \mathbb{R}^d$.*

Remark 5.16. *If $p > d/2$ then the limit exists a.s., if $p \leq d/2$ then far-away charges/stars have divergent effect.*

Remark 5.17. *Case of Coulomb or gravitational forces: In*

$$d \geq 3: \qquad p = d - 1 > d/2, \qquad \textit{OK}!$$

Towards more general limit theorems

Definition 5.3. *The function $L : (0, \infty) \to (0, \infty)$ is slowly varying (at infinity) iff*

$$(\forall a > 0): \qquad \lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1.$$

Examples, remarks, HWs

(1) If $\lim_{x\to\infty} L(x) = b \in (0, \infty)$ then obviously L is s.v.

(2) For any $\beta \in \mathbb{R}$, $L(x) := (\log x)^\beta$ is s.v.

(3) Show that for $\beta < 1$ and $c \in \mathbb{R}$, $L(x) := \exp\{c(\log x)^\beta\}$ is s.v.

(4) Construct a s.v. function L for which

$$\liminf_{x\to\infty} L(x) = 0, \quad \limsup_{x\to\infty} L(x) = \infty.$$

Theorem 5.6. *Let $X_1, X_2, \dots$ be i.i.d. with symmetric distribution F for which*

$$1 - F(x) = x^{-\alpha} L(x), \qquad \text{as} \quad x \to \infty,$$

where $\alpha \in (0, 2]$ and $L(x)$ is slowly varying at infinity. Let

$$a_n := \inf\{x : 1 - F(x) \leq 1/n\}.$$

Then

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\{iuS_n/a_n\}\right) = \exp\{-c|u|^\alpha\}$$

with some $c \in (0, \infty)$.

Remarks

Remark 5.18. *This extends (quite far) the previous limit theorems.*

Remark 5.19. *The proof is more technical. We omit it.*

5.6 Without symmetry

Stable distributions are parametrized by:

– the index $\alpha \in (0, 2]$;

– the skewness $\kappa \in [-1, 1]$;

– the scale $c \in (0, \infty)$;

– the shift $b \in \mathbb{R}$.

Remarks

Remark 5.20. *The scale and shift change with affine transformations. We will choose them later* $c = 1$, $b = 0$.

Remark 5.21. *The index and skewness are* relevant.

Remark 5.22. *Notation:* $STAB(\alpha, \kappa, c, b)$

Theorem 5.7. *The characteristic functions of stable distributions are*

$$\alpha \neq 1: \qquad \varphi(u) = \exp\left\{ibu - c|u|^{\alpha}\left(1 - i\,\mathrm{sgn}(u)\,\kappa\,\tan\frac{\alpha\pi}{2}\right)\right\}$$

$$\alpha = 1: \qquad \varphi(u) = \exp\left\{ibu - c|u|\left(1 + i\,\mathrm{sgn}(u)\,\kappa\,\frac{2\log|u|}{\pi}\right)\right\}.$$

Remarks

Remark 5.23. Symmetric *stable laws:* $\kappa = 0$.

Remark 5.24. *No skewness for* $\alpha = 2$.

Remark 5.25. *All stable laws are absolutely continuous with* C^{∞} *density. Follows from fast decay of the chf. as* $|u| \to \infty$.

Remark 5.26. *No explicit formula for the distribution/density function, except for the (already known) cases:*

- $\alpha = 2$ *(Gauss)*;
- $\alpha = 1, \kappa = 0$ *(Cauchy)*;
- $\alpha = 1/2, \kappa = \pm 1$ *(Lévy)*.

Remark 5.27. *Let* $X, Y \sim STAB(\alpha,\, \kappa = 1,\, c = 1,\, b = 0)$ *be i.i.d. and* $p,\, q \geq 0$ *such that* $p^{-\alpha} + q^{-\alpha} = 1$. *Then*

$$pX - qY \sim STAB(\alpha,\, \kappa = p^{-\alpha} - q^{-\alpha},\, c = 1,\, b = 0).$$

Remark 5.28. *"Heavy tail": From the type of singularity of $\varphi(u)$ at $u = 0$ it follows that*

$$(\forall \varepsilon > 0): \quad \mathbf{E}\left(|X|^{\alpha-\varepsilon}\right) < \infty,$$
$$(\forall \varepsilon \geq 0): \quad \mathbf{E}\left(|X|^{\alpha+\varepsilon}\right) = \infty.$$

More precisely:

$$\mathbf{P}\left(|X| > x\right) \sim Cx^{-\alpha}$$

Remark 5.29. "Lower tail" in the totally skew ($\kappa = 1$) case. *Fix:* $b = 0$ ($\checkmark$), $c = 1$ ($\checkmark$), $\kappa = 1$ (!)*: Then the chf*

$$\alpha \neq 1: \qquad \varphi(u) = \exp\Big\{ -|u|^\alpha \big(1 - i\ \mathrm{sgn}(u)\ \tan\frac{\alpha\pi}{2}\big)\Big\}$$
$$\alpha = 1: \qquad \varphi(u) = \exp\Big\{ -|u| \big(1 + i\ \mathrm{sgn}(u)\ \frac{2\log|u|}{\pi}\big)\Big\}.$$

can be continued analytically into the complex upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$*:, as* $\varphi_+ : \mathbb{C}_+ \to \mathbb{C}$*,*

$$\alpha \neq 1: \qquad \varphi_+(z) = \exp\Big\{ -\cos(\alpha\pi/2)^{-1}\,(-i\,z)^\alpha\Big\}$$
$$\alpha = 1: \qquad \varphi_+(z) = \exp\Big\{\big(i - \frac{2}{\pi}\log(z)\big)\,(i\,z)\Big\}$$

No analytic continuation (matching both halflines $u > 0$ and $u < 0$) into the lower half plane!

Theorem 5.8. *Let $X \sim STAB(\alpha, \kappa = +1, c = 1, b = 0)$. The "moment generating function" $\widetilde{\varphi} : [0, \infty) \to \mathbb{R}_+$*

$$\widetilde{\varphi}(u) := \mathbf{E}\left(\exp\{-uX\}\right) = \varphi_+(iu)$$

is

$$\alpha \neq 1: \qquad \widetilde{\varphi}(u) = \exp\Big\{ -\cos(\alpha\pi/2)^{-1}\,u^\alpha\Big\}$$
$$\alpha = 1: \qquad \widetilde{\varphi}(u) = \exp\Big\{\frac{2}{\pi}\,u\,\log(u)\Big\}.$$

Proof. Just done. □

Corollary 5.1. *(i) For* $\alpha \in (0,1)$*:*

$$\begin{aligned}&\mathbf{P}(X > 0) = 1,\\ &\frac{d}{dx}\mathbf{P}(0 < X < x) \sim c(\alpha)e^{-1/x}x^{-(\alpha+1)}, \quad \textit{as} \quad x \to 0.\end{aligned}$$

(ii) For $\alpha = 1$*:*

$$\mathbf{P}(X < -x) < \exp\{-ce^{x}\}$$

(iii) For $\alpha \in (1,2)$*:*

$$\mathbf{P}(X < -x) < \exp\{-cx^{\alpha/(\alpha-1)}\}$$

Proof. Tauberian arguments. . . . □

Theorem 5.9 (Limit theorem in the non-symmetric case). *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables. Assume*

(1) $\mathbf{P}(|X_j| > x) = x^{-\alpha}L(x)$ *with* $\alpha \in (0,2)$*,*

(2) $\lim_{x\to+\infty} \dfrac{\mathbf{P}(X_j > x)}{\mathbf{P}(X_j < -x)} =: \dfrac{1+\kappa}{1-\kappa} \in [0,\infty]$ *exists.*

Define

$$\begin{aligned}a_n &:= \inf\{x : \mathbf{P}(|X_j| > x) < n^{-1}\},\\ b_n &:= n\mathbf{E}\left(X_j 1\!\!1_{|X_j|\le a_n}\right).\end{aligned}$$

Then

$$\frac{S_n - b_n}{a_n} \Rightarrow STAB(\alpha,\kappa,c,b),$$

with some $c \in (0,\infty)$*,* $b \in \mathbb{R}$*.*

Remark 5.30. *Note that*

$$a_n = n^{1/\alpha}\widetilde{L}(n),$$

with $\widetilde{L}$ *slowly varying at infinity.*

Chapter 6

Infinitely divisible distributions

6.1 Infinite divisibility

Definition 6.1. *The probability distribution* F *is* *infinitely divisible* *iff for any* $n \in \mathbb{N}$ *there exists a probability distribution* F_n *so that*

$$F = \big(F_n\big)^{*n}.$$

Remarks

Remark 6.1. *In terms of the random variables:* X *is infinitely divisible iff for any* $n \in \mathbb{N}$ *there are* $X_{n,1}, X_{n,2}, \dots, X_{n,n}$ *i.i.d. so that*

$$X \sim X_{n,1} + X_{n,2} + \cdots + X_{n,n}.$$

Remark 6.2. *In terms of the characteristic functions:* $\varphi(u)$ *is an infinitely divisible chf. iff for any* $n \in \mathbb{N}$ *there exists a chf.* $\varphi_n(u)$ *so that*

$$\varphi(u) = \big(\varphi_n(u)\big)^n.$$

6.2 Examples

Example 6.1. *The normal distribution:*

$$N(0,\sigma^2=t)=N(0,\sigma^2=t/n)^{*n}.$$

$$f_t(y)=\frac{1}{\sqrt{2\pi t}}e^{-y^2/(2t)}, \qquad \varphi_t(u)=\exp\{-tu^2/2\}$$

Example 6.2. *The Cauchy distributions:*

$$CAU(0,\tau=t)=N(0,\tau=t/n)^{*n}.$$

$$f_t(y)=\frac{1}{\pi}\frac{t}{t^2+y^2}, \qquad \varphi_t(u)=\exp\{-t|u|\}$$

Example 6.3. *Stable distributions in general: If*

$$X\sim STAB(\alpha,\kappa,c=t,b=0),$$

and

$$X_{n,1},\dots,X_{n,n}\sim STAB(\alpha,\kappa,c=t/n,b=0)$$

are i.i.d. Then

$$X\sim n^{-\alpha}\big(X_{n,1}+\dots+X_{n,n}\big)=\frac{X_{n,1}}{n^\alpha}+\dots+\frac{X_{n,n}}{n^\alpha}.$$

The density and characteristic functions (for $\alpha\neq 1$):

$$f_t(y)=???, \qquad \varphi_t(u)=\exp\Big\{-t|u|^\alpha\big(1-i\ \mathrm{sgn}(u)\,\kappa\,\tan\frac{\alpha\pi}{2}\big)\Big\}.$$

Example 6.4. *Poisson (Not stable!):*

$$POI(\varrho t) = POI(\varrho t/n)^{*n}.$$

$$p_t(k) = e^{-t\rho}\frac{(t\rho)^k}{k!}, \qquad \varphi_t(u) = \exp\left\{t\rho(e^{iu}-1)\right\}$$

Example 6.5. *The gamma distributions (Not stable!):*

$$GAM(t) = GAM(t/n)^{*n}.$$
$$f_t(y) = \Gamma(t)^{-1}e^{-y}y^{t-1}, \qquad \varphi_t(u) = \exp\left\{-t\log(1-iu)\right\}.$$

Example 6.6. *The negative binomial distributions (Not stable!):*

$$NB(p,t) = NB(p,t/n)^{*n}.$$

$$p_t(k) = (-1)^k\binom{-t}{k}(1-p)^t p^k, \qquad \varphi_t(u) = \exp\left\{-t\log\frac{1-pe^{iu}}{1-p}\right\}.$$

where, for $r \in \mathbb{R}$

$$\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!}, \qquad \sum_{k=0}^{\infty}\binom{r}{k}x^k = (1+x)^r.$$

Example 6.7. *(a) The compound Poisson distribution (CPOI): Let $\xi_1, \xi_2 \dots$ be i.i.d. with distribution*

$$\mathbf{P}(\xi_j < x) = G(x),$$

and $\nu \sim POI(\varrho)$ independent of the ξ_j-s. Then we call

$$X := \sum_{j=1}^{\nu} \xi_j \sim CPOI(G, \varrho).$$

Then:

$$CPOI(G, \varrho t) = CPOI(G, \varrho t/n)^{*n}$$

follows from infinite divisibility of Poisson: Let N_t be a Poisson process of intensity $\varrho > 0$, independent of the ξ_j-s and

$$X_t := \sum_{j=1}^{N_t} \xi_j = \sum_{m=1}^{n} \big(X_{mt/n} - X_{(m-1)t/n}\big).$$

where $X_{mt/n} - X_{(m-1)t/n}$, $m = 1, 2, \dots, n$, are i.i.d.

The characteristic function of $CPOI(G, \varrho t)$ is

$$\begin{aligned}\varphi_t(u) &= \mathbf{E}(\exp\{iuX_t\}) \\ &= \sum_{m=0}^{\infty} e^{-\varrho t} \frac{(\varrho t)^m}{m!} \Big(\int_{-\infty}^{\infty} e^{iuy} dG(y) \Big)^m \\ &= \exp\Big\{ \varrho t \int_{-\infty}^{\infty} (e^{iuy} - 1) dG(y) \Big\}.\end{aligned}$$

(b) The centred compound Poisson distribution (CCPOI): Let $\xi_1, \xi_2 \dots$ be i.i.d. with distribution

$$\mathbf{P}(\xi_j < x) = G(x),$$

and $\nu \sim POI(\varrho)$ independent of the ξ_j-s. Then we call

$$\widetilde{X} := \sum_{j=1}^{\nu} \xi_j - \varrho \mathbf{E}(\xi_j) \sim CCPOI(G, \varrho).$$

Remark 6.3. *Mind that* $CCPOI(G, \varrho) \neq CPOI(\widetilde{G}, \varrho)$*! (Here* $\widetilde{G}$ *is the centred distribution.) Then:*

$$CCPOI(G, \varrho t) = CCPOI(G, \varrho t/n)^{*n}$$

follows from infinite divisibility of Poisson:

Let N_t *be a Poisson process of intensity* $\varrho > 0$*, independent of the* ξ_j*-s and*

$$\widetilde{X}_t := \sum_{j=1}^{N_t} \xi_j - \varrho t \mathbf{E}(\xi) = \sum_{m=1}^{n} \big(\widetilde{X}_{mt/n} - \widetilde{X}_{(m-1)t/n}\big).$$

where $\widetilde{X}_{mt/n} - \widetilde{X}_{(m-1)t/n}$*,* $m = 1, 2, \ldots, n$*, are i.i.d.*

The characteristic function of $CCPOI(G, \varrho t)$ *is*

$$\begin{aligned}
\varphi_t(u) &= \mathbf{E}\left(\exp\{iu\widetilde{X}_t\}\right) \\
&= \sum_{m=0}^{\infty} e^{-\varrho t} \frac{(\varrho t)^m}{m!} \Big(\int_{-\infty}^{\infty} e^{iuy} dG(y)\Big)^m \exp\big\{ -iu\varrho t \mathbf{E}(\xi) \big\} \\
&= \exp\Big\{\varrho t \int_{-\infty}^{\infty} (e^{iuy} - 1 - iuy) dG(y)\Big\}.
\end{aligned}$$

Remarks

Remark 6.4. *In all examples we have seen a one parameter family of random variables* $\big(X_t\big)_{t\geq 0}$ *such that for any* $n \in \mathbb{N}$

$$X_t \sim \sum_{m=1}^{n} X'_{t/n,m}$$

where $X'_{t/n,1}, X'_{t/n,2}, \ldots, X'_{t/n,n} \sim X_{t/n}$ *are i.i.d.*

It is reasonable to expect the existence of a process $t \mapsto X_t$ *with stationary and independent increments and* $X_0 = 0$ *= a Lévy process.*

[Mind the difference between a one-parameter family of random variables and a process: process = consistent family of joint distributions of finite dimensional marginals.]

- Example 6.1: $t \mapsto X_t$ is standard Brownian motion.

- Example 6.2: The Cauchy process: Let (ξ_s, η_s) two-dim. Brownian motion, $(\xi_0, \eta_0) = (0,0)$. Let

$$\tau_t := \inf\{s : \xi_s = t\}, \qquad X_t := \eta_{\tau_t}.$$

- Example 6.3: Stable processes. E.g. for $\alpha = \frac{1}{2}, \kappa = 1$:

$$X_t := \tau_t \quad \text{of the previous example.}$$

- Example 6.4: X_t is the Poisson process.
- Example 6.5, Example 6.6: later
- Example 6.7: Defined from start as a process: the compound Poisson process.

Remark 6.5. *If X and Y are infinitely divisible and independent then $Z := X + Y$ is infinitely divisible. (HW!) If X_t and Y_t are two independent Lévy processes then $Z_t := X_t + Y_t$ is also a Lévy process. (HW!)*

Lemma 6.1. *If X is infinitely divisible and $\varphi(u) := \mathbf{E}(\exp\{iuX\})$ then $(\forall u \in \mathbb{R}) : \varphi(u) \neq 0$.*

Proof. Let $X_{n,1}, \dots, X_{n,n}$ be i.i.d. so that $X \sim X_{n,1} + \dots + X_{n,n}$. Then $X_{n,1} \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$ (HW!). It follows that for any $u \in \mathbb{R}$

$$\lim_{n\to\infty} \varphi(u)^{1/n} = \lim_{n\to\infty} \mathbf{E}(\exp\{iuX_{n,1}\}) = 1.$$

□

Thus: $\psi : \mathbb{R} \to \mathbb{C}$, $\psi(u) := \log \varphi(u)$ is well defined.

Remark 6.6. *If $t \mapsto X_t$ is a Lévy process with $X_1 \sim X$, then*

$$\varphi_t(u) := \mathbf{E}(\exp\{iuX_t\}) = \exp\{t\psi(u)\}.$$

Remark 6.7. *If X is infinitely divisible and $\varphi(u) := \mathbf{E}\left(\exp\{iuX\}\right)$ then for any $a > 0$ and $\beta > 0$:*

$$\widetilde{\varphi}(u) := \big(\varphi(au)\big)^{\beta}$$

is infinitely divisible chf.

If $t \mapsto X_t$ is a Lévy process then so is $t \mapsto \widetilde{X}_t := aX_{\beta t}$ and

$$\begin{aligned}\widetilde{\varphi}_t(u) &= \mathbf{E}\left(\exp\{iu\widetilde{X}_t\}\right) = \mathbf{E}\left(\exp\{iuaX_{\beta t}\}\right) \\ &= \exp\{\beta t\psi(au)\} = \big(\varphi_t(au)\big)^{\beta}.\end{aligned}$$

Remark 6.8. *If X_n, $n = 1, 2 \ldots$ are infinitely divisible and $X_n \Rightarrow X$ then X is also infinitely divisible. (HW!)*

Remark 6.9. *If X is infinitely divisible then for any $K < \infty$: $\mathbf{P}\left(|X| > K\right) > 0$. (HW!)*

6.3 Back to the examples

All the previous examples are derived in some way from the compound Poisson.

- Example 6.1, normal: $\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with distribution

  $$\mathbf{P}\left(\xi_j^{(\varepsilon)} = \pm\varepsilon\right) = 1/2.$$

  $N_t^{(\varepsilon)}$ Poisson process of intensity ε^{-2}, independent of the $\xi_j^{(\varepsilon)}$-s. *CPOI* process and its chf.:

  $$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\left(\exp\{iuX_t^{(\varepsilon)}\}\right).$$

  Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

  $$\psi^{(\varepsilon)}(u) = \varepsilon^{-2}(\cos(\varepsilon u) - 1) \to -\frac{u^2}{2}.$$

- Example 6.2, symmetric stable of index $\alpha \in (0,2)$: $\xi_j^{(\varepsilon)}$, $j = 1,2,\ldots$ i.i.d. with symmetric distribution

$$\mathbf{P}\left(|\xi_j^{(\varepsilon)}| > |y|\right) = \min\{(|y|/\varepsilon)^{-\alpha}, 1\}$$
$$\frac{d}{dy}\mathbf{P}\left(\xi_j^{(\varepsilon)} < y\right) = \frac{1}{2}\alpha\varepsilon^{\alpha}|y|^{-\alpha-1}1\!\!1_{|y|>\varepsilon}.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.

$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\left(\exp\{iuX_t^{(\varepsilon)}\}\right).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}\psi^{(\varepsilon)}(u) &= \varepsilon^{-\alpha}\int_{|y|>\varepsilon}(e^{iuy}-1)\frac{1}{2}\alpha\varepsilon^{\alpha}|y|^{-\alpha-1}dy\\ &= \alpha\int_{\varepsilon}^{\infty}(\cos(uy)-1)y^{-\alpha-1}dy\\ &\xrightarrow{(1)} \alpha\int_0^{\infty}(\cos(uy)-1)y^{-\alpha-1}dy = -c\,|u|^{\alpha},\end{aligned}$$

where

$$c := \alpha\int_0^{\infty}\frac{1-\cos y}{y^{\alpha+1}}dy.$$

(1) : absolute integrability at 0 and at ∞.

- Example 6.3, skew stable: later,
- Example 6.4, Poisson: nothing to prove,
- Example 6.5, gamma: $\xi_j^{(\varepsilon)}$, $j = 1,2,\ldots$ i.i.d. with distribution density

$$\frac{d}{dy}\mathbf{P}\left(\xi_j^{(\varepsilon)} < y\right) = \frac{1}{\varrho(\varepsilon)}\frac{e^{-y}}{y}1\!\!1_{y>\varepsilon}, \quad \text{with} \quad \varrho(\varepsilon) := \int_{\varepsilon}^{\infty}\frac{e^{-y}}{y}dy.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varrho(\varepsilon)$, independent of the $\xi_j^{(\varepsilon)}$-s.
$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\left(\exp\{iuX_t^{(\varepsilon)}\}\right).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}
\psi^{(\varepsilon)}(u) &= \varrho(\varepsilon)\int_\varepsilon^\infty \left(e^{iuy}-1\right)\varrho(\varepsilon)^{-1}\frac{e^{-y}}{y}dy \\
&= \int_\varepsilon^\infty \left(e^{iuy}-1\right)\frac{e^{-y}}{y}dy \\
&\xrightarrow{(1)} \int_0^\infty (e^{iuy}-1)\frac{e^{-y}}{y}dy \overset{(2)}{=} -\log(1-iu)
\end{aligned}$$

(1) : absolute integrability at 0.

(2) : HW!

- Example 6.6, negative binomial: HW: Construct $NB(p,t)$ as compound Poisson.

- (a) Example 6.3, *skew stable,* $STAB(\alpha,\kappa=1,c,0)$, *with* $\alpha\in(0,1)$: $\xi_j^{(\varepsilon)}$, $j=1,2,\dots$ i.i.d. with distribution

$$\begin{aligned}
\mathbf{P}\left(\xi_j^{(\varepsilon)}>y\right) &= \min\{(y/\varepsilon)^{-\alpha},1\}, & y>0,\\
\frac{d}{dy}\mathbf{P}\left(\xi_j^{(\varepsilon)}<y\right) &= \alpha\varepsilon^\alpha|y|^{-\alpha-1}1\!\!1_{y>\varepsilon}, & y>0.
\end{aligned}$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.
$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\left(\exp\{iuX_t^{(\varepsilon)}\}\right).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}
\psi^{(\varepsilon)}(u) &= \varepsilon^{-\alpha}\int_{\varepsilon}^{\infty}(e^{iuy}-1)\alpha\varepsilon^{\alpha}y^{-\alpha-1}dy \\
&= \alpha\int_{\varepsilon}^{\infty}(e^{iuy}-1)y^{-\alpha-1}dy \\
&\xrightarrow{(1)} \alpha\int_{0}^{\infty}(e^{iuy}-1)y^{-\alpha-1}dy \\
&\stackrel{(2)}{=} -\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\big(1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(u)\big)\,|u|^{\alpha}.
\end{aligned}$$

(1) : The real part is absolutely integrable at ∞ and at 0 for any $\alpha\in(0,2)$ ✓.
The imaginary part is absolutely integrable at ∞ for any $\alpha\in(0,2)$ ✓, but at 0 only for $\alpha\in(0,1)$!!!

(2) Computation below.

$$\begin{aligned}
&\alpha\int_{0}^{\infty}(e^{iy}-1)y^{-\alpha-1}dy \\
&\quad\stackrel{(1)}{=} \lim_{\varepsilon\to 0}\alpha\int_{0}^{\infty}(e^{-(\varepsilon-i)y}-1)y^{-\alpha-1}dy \\
&\quad= -\lim_{\varepsilon\to 0}(\varepsilon-i)\alpha\int_{0}^{\infty}\big(\int_{0}^{y}e^{-(\varepsilon-i)z}dz\big)y^{-\alpha-1}dy \\
&\quad\stackrel{(2)}{=} -\lim_{\varepsilon\to 0}(\varepsilon-i)\int_{0}^{\infty}\big(\alpha\int_{z}^{\infty}y^{-\alpha-1}dy\big)e^{-(\varepsilon-i)z}dz \\
&\quad= -\lim_{\varepsilon\to 0}(\varepsilon-i)\int_{0}^{\infty}e^{-(\varepsilon-i)z}z^{-\alpha}dz \\
&\quad\stackrel{(3)}{=} -\lim_{\varepsilon\to 0}(\varepsilon-i)^{\alpha}\int_{0}^{\infty}e^{-z}z^{-\alpha}dz \\
&\quad= -\Gamma(1-\alpha)e^{-i\pi\alpha/2}.
\end{aligned}$$

(1) : DC, valid only for $\alpha\in(0,1)$!!!;

(2) : Fubini;

(3) : Change of integration path in $\mathbb{C}$ (HW!).

(b) Example 6.3, *skew stable*, $STAB(\alpha, \kappa = 1, c, 0)$, *with* $\alpha \in (1, 2)$:
Centre! $\xi_j^{(\varepsilon)}$, $j = 1, 2, \dots$ i.i.d. with distribution

$$\mathbf{P}\left(\xi_j^{(\varepsilon)} > y\right) = \min\{(y/\varepsilon)^{-\alpha}, 1\}, \qquad y > 0,$$

$$\frac{d}{dy}\mathbf{P}\left(\xi_j^{(\varepsilon)} < y\right) = \alpha\varepsilon^{\alpha}|y|^{-\alpha-1}\mathbb{1}_{y>\varepsilon}, \qquad y > 0.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.
$CCPOI$ process and its chf.:

$$\widetilde{X}_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)} - \mathbf{E}\left(N_t^{(\varepsilon)}\right)\mathbf{E}\left(\xi_j^{(\varepsilon)}\right),$$

$$\exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\left(\exp\{iu\widetilde{X}_t^{(\varepsilon)}\}\right).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}
\psi^{(\varepsilon)}(u) &= \varepsilon^{-\alpha}\int_{\varepsilon}^{\infty}(e^{iuy} - 1 - iuy)\alpha\varepsilon^{\alpha}y^{-\alpha-1}dy \\
&= \alpha\int_{\varepsilon}^{\infty}(e^{iuy} - 1 - iuy)y^{-\alpha-1}dy \\
&\xrightarrow{(1)} \alpha\int_{0}^{\infty}(e^{iuy} - 1 - iuy)y^{-\alpha-1}dy \\
&\stackrel{(2)}{=} \frac{\Gamma(2-\alpha)}{\alpha-1}\cos\frac{\pi\alpha}{2}\left(1 - i\tan\frac{\pi\alpha}{2}\,\mathrm{sgn}(u)\right)|u|^{\alpha}.
\end{aligned}$$

(1) : The real part is absolutely integrable at 0 and at ∞ for any $\alpha \in (0, 2)$ ✓.
The centred imaginary part is absolutely integrable at 0 for any $\alpha \in (0, 2)$ ✓, but at ∞ only for $\alpha \in (1, 2)$!!!

(2) : Computation below.

$$
\begin{aligned}
\alpha \int_0^\infty & (e^{iy} - 1 - iy) y^{-\alpha-1} dy \\
&= i\alpha \int_0^\infty \Big(\int_0^y (e^{iz} - 1) dz \Big) y^{-\alpha-1} dy \\
&\overset{(1)}{=} i \int_0^\infty \Big(\alpha \int_z^\infty y^{-\alpha-1} dy \Big) (e^{iz} - 1) dz \\
&= i \int_0^\infty (e^{iz} - 1) z^{-\alpha} dz \\
&\overset{(2)}{=} \frac{-1}{\alpha - 1} \int_0^\infty e^{iz} z^{1-\alpha} dz \\
&\overset{(3)}{=} \frac{i^{-\alpha}}{\alpha - 1} \int_0^\infty e^{-z} z^{1-\alpha} dz \\
&= \Gamma(2-\alpha)/(\alpha - 1) e^{-i\pi\alpha/2}.
\end{aligned}
$$

(1) : Fubini;
(2) : integration by parts;
(3) : Change of integration path in $\mathbb{C}$ (HW!).

Definition 6.2. *A non-negative sigma-finite measure on* $\mathbb{R}$ *for which*

(1) $\mu((-\infty, -1] \cup [1, \infty)) < \infty$*,*

(2) $\int y^2 \mathbb{1}_{|y|<1} d\mu(y) < \infty$*,*

(3) $\mu(0) = 0$

is called Lévy measure.

Theorem 6.1 (Aleksandr Yakovlevich Khinchin, Paul Lévy). *Characteristic functions of infinitely divisible distributions are exactly the functions of the form* $\varphi_t(u) = \exp\{t\psi(u)\}$ *with*

$$
\psi(u) = ibu - \frac{\sigma^2}{2} u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1 - iuy \mathbb{1}_{|y|<1} \big) d\mu(y), \qquad \text{(LH)}
$$

where $b \in \mathbb{R}$*,* $\sigma^2 \geq 0$*, and* μ *is a Lévy measure.*

Remarks

Remark 6.10. *The parameters b, σ^2 and μ are uniquely determined.*

Remark 6.11. *(LH) is the Lévy–Khinchin formula.*

Remark 6.12. *If $g : \mathbb{R} \to \mathbb{R}$ is such that*

$$\int_{\mathbb{R}} \left| y 1\!\!1_{|y|<1} - g(y) \right| d\mu(y) < \infty$$

then the Lévy–Khinchin formula can be written

$$\psi(u) = ib'u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1 - iug(y)\big) d\mu(y),$$

with

$$b' = b - \int_{\mathbb{R}} \big(y 1\!\!1_{|y|<1} - g(y)\big) d\mu(y)$$

Remark 6.13. *Another usual conventional choice*

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1 - iu\frac{y}{1+y^2}\big) d\mu(y),$$

Remark 6.14. *If*

$$\int_{\mathbb{R}} |y| d\mu(y) < \infty$$

then the Lévy–Khinchin formula can be written

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1\big) d\mu(y),$$

Proof. We prove that functions $\varphi_t(u) = \exp\{t\psi(u)\}$ with $\psi(u)$ given by (LH) are indeed chf-s of infinitely divisible distributions. We write $\psi(u) = \psi_1(u)+$

$\psi_2(u) + \psi_2(u) + \psi_4(u)$, with

$$\psi_1(u) = ibu, \qquad \psi_2(u) = -\frac{\sigma^2}{2}u^2$$
$$\psi_3(u) = \int_{\mathbb{R}} \left(e^{iuy} - 1\right) 1\!\!1_{|y|\geq 1} d\mu(y)$$
$$\psi_4(u) = \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{|y|<1} d\mu(y)$$

Then:

– $\psi_1(u)$ comes from a simple shift by b.

– $\psi_2(u)$ comes from a Gaussian $\sim N(0, \sigma^2)$.

– $\psi_3(u)$ comes from a compound Poisson $CPOI(\varrho, F)$ with

$$\varrho = \mu(\{y : |y| \geq 1\}), \qquad dF(y) = \varrho^{-1} 1\!\!1_{|y|\geq 1} d\mu(y).$$

– $\psi_4(u)$ comes as weak limit of a sequence of $CCPOI(\varrho^{(\varepsilon)}, G^{(\varepsilon)})$:

$$\begin{aligned}
&\int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{|y|<1} d\mu(y) \\
&\qquad = \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{\varepsilon<|y|<1} d\mu(y) \\
&\qquad = \lim_{\varepsilon\to 0} \varrho^{(\varepsilon)} \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{\varepsilon<|y|<1} dG^{(\varepsilon)}(y)
\end{aligned}$$

where

$$\varrho^{(\varepsilon)} := \mu(\{y : \varepsilon < |y| < 1\}), \qquad dG^{(\varepsilon)}(y) = (\varrho^{(\varepsilon)})^{-1} 1\!\!1_{\varepsilon<|y|<1} d\mu(y).$$

□

6.4 Lévy measure of stable laws

For $STAB(\alpha,\kappa,c,b)$

$$\psi(u)=ibu+c\int_{-\infty}^{\infty}\big(e^{iuy}-1-iuy\mathbb{1}_{|y|<1}\big)d\mu_{\alpha,\kappa}(y),$$
$$d\mu_{\alpha,\kappa}(y):=\left(\frac{1+\kappa}{2}\mathbb{1}_{y>0}+\frac{1-\kappa}{2}\mathbb{1}_{y<0}\right)\frac{1}{|y|^{\alpha+1}}dy.$$

Remark 6.15. *These are the only homogeneous Lévy measures.*

Remark 6.16. *Alternative forms:*

$$\alpha\in(0,1):\qquad \psi(u)=ib'u+c\int_{-\infty}^{\infty}\big(e^{iuy}-1\big)d\mu_{\alpha,\kappa}(y),$$
$$\alpha\in(1,2):\qquad \psi(u)=ib'u+c\int_{-\infty}^{\infty}\big(e^{iuy}-1-iuy\big)d\mu_{\alpha,\kappa}(y),$$

Proof. If

$$\psi(u)=ibu-\frac{\sigma^2}{2}u^2+\int_{-\infty}^{\infty}\big(e^{iuy}-1-iuy\mathbb{1}_{|y|<1}\big)d\mu(y),$$

and $a>0$ then

$$\widetilde{\psi}(u):=\psi(au)=i\widetilde{b}u-\frac{\widetilde{\sigma}^2}{2}u^2+\int_{-\infty}^{\infty}\big(e^{iuy}-1-iuy\mathbb{1}_{|y|<1}\big)d\widetilde{\mu}(y),$$

with

$$\widetilde{b}=ab-a\int y\big(\mathbb{1}_{|y|<1}-\mathbb{1}_{|y|<a^{-1}}\big)d\mu(y),\quad \widetilde{\sigma}^2=a^2\sigma^2,\quad d\widetilde{\mu}(y)=d\mu(y/a).$$

Stability: $(\forall\ a_1,a_2>0)$ $(\exists\ a_3>0,b_3\in\mathbb{R})$ such that:

$$\psi(a_1u)+\psi(a_2u)=ib_3u+\psi(a_3u).$$

It follows that

$$d\mu(y/a_1) + d\mu(y/a_2) = d\mu(y/a_3), \tag{6.1}$$
$$(a_1^2 + a_2^2)\sigma^2 = a_3^2\sigma^2. \tag{6.2}$$

(6.1) implies homogeneity of μ: with some $\alpha \in (0, 2)$ and $C_+, C_- \geq 0$

$$d\mu(y) = \big(C_+ 1\!\!1_{y>0} + C_- 1\!\!1_{y<0}\big) \frac{1}{|y|^{\alpha+1}} dy,$$

and

$$\{ C_+ + C_- > 0 \} \quad \Longrightarrow \quad \{ a_1^\alpha + a_2^\alpha = a_3^\alpha. \}.$$

From (6.2) it follows that either $\sigma^2 = 0$ or $C_+ = 0 = C_-$. □

6.4.1 Poisson point processes

Let (S, d) be a complete, separable, locally compact metric space. E.g. $S = \mathbb{R}^n$, or $S = (-\infty, 0) \cup (0, \infty)$ with properly chosen metrization.

The space of locally finite point systems:

$$\Pi = \Pi(S) := \big\{ \mathcal{X} \subset S \,:\, (\forall K \Subset S) : |\mathcal{X} \cap K| < \infty \big\}.$$

$\Pi(S)$ is endowed with a natural metric topology, ..., Borel sigma algebra $\mathcal{F}$.

Counting functions: for $K \Subset S$

$$m_K : \Pi \to \mathbb{N}, \qquad m_K(\mathcal{X}) := |\mathcal{X} \cap K|.$$

Definition 6.3. *A (random) point process on* (S, d) *is a* $(\Pi, \mathcal{F})$*-valued random variable,* Ξ*. I.e. it is a probability measure on* $(\Pi, \mathcal{F})$*.*

Definition 6.4. *Let* μ *be a sigma-finite, tight positive measure on* S*. The Poisson point process with intensity mesure* μ *– denoted* $PPP(\mu)$ *– is* the unique *point process* Ξ *on* (S, d) *satisfying the following: If* $K_1, \ldots, K_n \Subset S$ *are disjoint then* $\big(m_{K_1}(\Xi), \ldots, m_{K_n}(\Xi)\big)$ *are independent, and* $m_{K_j}(\Xi) \sim POI(\mu(K_i))$*.*

Remark 6.17. *Existence of $PPP(\mu)$: see constructions below. Uniqueness of $PPP(\mu)$:*

- Construction for $\mu(S) < \infty$: *Let $\xi_1, \xi_2, \dots \in S$ be i.i.d., and $\nu \in \mathbb{N}$ independent of the ξ_j-s, with distribution*

$$\mathbf{P}(\xi_j \in A) = \frac{\mu(A)}{\mu(S)}, \qquad \nu \sim POI(\mu(S)).$$

  Then $\Xi := \{\xi_1, \xi_2, \dots, \xi_\nu\}$ is $PPP(\mu)$. (HW!).

- Construction for $\mu(S) = \infty$: *Let $S = \cup_{k=1}^{\infty} S_k$, with disjoint S_k-s and $(\forall k) : \mu(S_k) < \infty$. Let $\mu_k(\cdot) := \mu(\cdot \cap S_k)$, and $\Xi_k \sim PPP(\mu_k)$ as defined above. Then $\Xi := \bigcup_{k=1}^{\infty} \Xi_k$ is $PPP(\mu)$. (HW!).*

Remarks

Remark 6.18. *A theorem of Rényi:*

Theorem 6.2. *Let μ be a non-atomic measure on S. (That is: $(\forall x \in S) : \mu(\{x\}) = 0$.) Let $\mathcal{A} \subset \mathcal{P}(S)$ generate the Borel-algebra of (S, d). If for a point process $\mathcal{X}$ the following holds:*

$$(\forall A \in \mathcal{A}, \quad \text{with } \mu(A) < \infty) : \qquad m_A(\mathcal{X}) \sim POI(\mu(A)),$$

then $\mathcal{X} \sim PPP(\mu)$.

Homework 6.1. *Give counterexample with atomic μ!*

Remark 6.19. *Relation to $CPOI(\varrho, G)$: For $S = \mathbb{R}$, $\varrho := \mu(S) < \infty$, $dG(y) := \varrho^{-1} d\mu(y)$:*

$$\sum_{\xi \in \Xi} \xi =: X \sim CPOI(\varrho, G).$$

Remark 6.20. *Relation to Lévy measure, Lévy–Khinchin formula – summable case:* $S = (0, \infty)$. *If*

$$\text{(1):} \quad \int_1^\infty d\mu(y) < \infty, \qquad \text{(2):} \quad \int_0^1 y d\mu(y) < \infty,$$

let

$$X_1 := \sum_{\xi \in \Xi \cap [1,\infty)} \xi, \qquad X_2 := \sum_{\xi \in \Xi \cap (0,1)} \xi,$$

Then

$$\mathbf{P}(X_1 < \infty) = 1 \qquad (\checkmark),$$

$$\mathbf{E}(X_2) = \int_0^1 y d\mu(y) < \infty. \qquad \text{(HW!)}$$

The characteristic function of $X := X_1 + X_2$ *is:*

$$\mathbf{E}(\exp\{iuX\}) = \exp\Big\{\int_0^\infty (e^{iuy} - 1) d\mu(y)\Big\}.$$

Remark 6.21. *Relation to Lévy measure, Lévy–Khinchin formula – non-summable case:* $S = (0, \infty)$. *If*

$$\text{(1):} \quad \int_1^\infty d\mu(y) < \infty, \qquad \text{(2):} \quad \int_0^1 y^2 d\mu(y) < \infty,$$

let

$$X_1 := \sum_{\xi \in \Xi \cap [1,\infty)} \xi, \qquad X_{2,\varepsilon} := \sum_{\xi \in \Xi \cap (\varepsilon,1)} \xi - \int_\varepsilon^1 y d\mu(y).$$

Then

$$\mathbf{P}(X_1 < \infty) = 1 \qquad (\checkmark),$$
$$\mathbf{P}\Big(\exists \lim_{\varepsilon \to 0} X_{2,\varepsilon} =: X_2\Big) = 1, \qquad (HW!)$$

(Hint: Compute $\mathbf{Var}(X_{2,\varepsilon})$ *and use Kolmogorov's criterion.)*
The characteristic function is of $X := X_1 + X_2$ *is:*

$$\mathbf{E}(\exp\{iuX\}) = \exp\Big\{\int_0^\infty (e^{iuy} - 1 - iuy 1\!\!1_{0<y<1}) d\mu(y)\Big\}.$$

6.4.2 Back to stable convergence

Definition 6.5. *The function* $L : (0, \infty) \to (0, \infty)$ *is slowly varying (at infinity) iff*

$$(\forall a > 0): \qquad \lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1.$$

Examples, remarks, HWs

(1) If $\lim_{x\to\infty} L(x) = b \in (0, \infty)$ then obviously L is s.v.

(2) For any $\beta \in \mathbb{R}$, $L(x) := (\log x)^\beta$ is s.v.

(3) Show that for $\beta < 1$ and $c \in \mathbb{R}$, $L(x) := \exp\{c(\log x)^\beta\}$ is s.v.

(4) Construct a s.v. function L for which

$$\liminf_{x\to\infty} L(x) = 0, \qquad \limsup_{x\to\infty} L(x) = \infty.$$

Definition 6.6. *The function $U : (0,\infty) \to (0,\infty)$ is regularly varying (at infinity) iff*

$$(\forall a > 0): \qquad \lim_{x\to\infty} \frac{U(ax)}{U(x)} \qquad \text{exists}$$

Fact 6.1. *The function $x \mapsto U(x)$ is regularly varying at ∞ if and only if $U(x) = x^{\beta} L(x)$ with some $\beta \in \mathbb{R}$ and $L(x)$ slowly varying. (HW!)*

Some basic facts about slowly varying functions

(1) If $\beta > -1$ then

$$\int_1^x y^{\beta} L(y) dy = \left(\frac{1}{\beta+1} + o(1) \right) x^{\beta+1} L(x).$$

(2) If $\beta < -1$ then

$$\int_x^{\infty} y^{\beta} L(y) dy = -\left(\frac{1}{\beta+1} + o(1) \right) x^{\beta+1} L(x).$$

(3)

$$L(x) = a(x) \exp\Big\{ \int_1^x \frac{\epsilon(y)}{y} dy \Big\}$$

$$\text{where} \qquad \exists \lim_{y\to\infty} a(y) =: c, \qquad \lim_{y\to\infty} \epsilon(y) = 0.$$

Theorem 6.3 (Skew stable limit theorem.). *Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables. Assume*

(1) $\mathbf{P}(|\xi_1| > x) = x^{-\alpha}L(x)$ *with* $\alpha \in (0,1) \cup (1,2)$,

(2) $\exists \lim_{x\to+\infty} \dfrac{\mathbf{P}(\xi_1 > x)}{\mathbf{P}(\xi_1 < -x)} =: \dfrac{1+\kappa}{1-\kappa} \in [0,\infty]$.

Define

$$a_n := \inf\{x \,:\, \mathbf{P}(|\xi_1| > x) < n^{-1}\} = n^{1/\alpha}\widetilde{L}(n).$$

(i) $\alpha \in (0,1)$ *case:*

$$\mathbf{E}\left(\exp\{iuS_n/a_n\}\right) \to \exp\Big\{\int_{\mathbb{R}} \big(e^{iuy} - 1\big) d\mu_{\alpha,\kappa}(y)\Big\}.$$

(ii) $\alpha \in (1,2)$ *case:*

$$\mathbf{E}\left(\exp\{iu(S_n - n\mathbf{E}(\xi_1))/a_n\}\right) \to \exp\Big\{\int_{\mathbb{R}} \big(e^{iuy} - 1 - iuy\big) d\mu_{\alpha,\kappa}(y)\Big\}.$$

Proof.

Lemma 6.2 (1).

$$\Big\{\frac{\xi_1}{a_n}, \frac{\xi_2}{a_n}, \dots, \frac{\xi_n}{a_n}\Big\} =: \Xi_n \Rightarrow PPP(\mu_{\alpha,\kappa}).$$

Lemma 6.3 (2). *Let X_n, $n = 1, 2, \dots,$ be a sequence of random variables and assume that for any $r = 1, 2, \dots,$ X_n is decomposed as $X_n = Y_{n,r} + Z_{n,r}$. If*

$$Y_{n,r} \overset{n\to\infty}{\Longrightarrow} Y_{\infty,r} \overset{r\to\infty}{\Longrightarrow} Y,$$

and

$$(\forall \delta > 0): \qquad \lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{P}(|Z_{n,r}| \ge \delta) = 0,$$

Then $X_n \overset{n\to\infty}{\Longrightarrow} Y$.

Case $\alpha \in (0,1)$

$$\begin{aligned} X_n &:= \frac{S_n}{a_n} = \sum_{j=1}^{n} \frac{\xi_j}{a_n} \\ &= \sum_{j=1}^{n} \frac{\xi_j}{a_n} \mathbb{1}_{\frac{|\xi_j|}{a_n} \geq r^{-1}} + \sum_{j=1}^{n} \frac{\xi_j}{a_n} \mathbb{1}_{\frac{|\xi_j|}{a_n} < r^{-1}} \\ &=: Y_{n,r} + Z_{n,r}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}\left(e^{iuY_{n,r}}\right) &\overset{n\to\infty}{\longrightarrow} \exp\Big\{ \int \left(e^{iuy} - 1\right) \mathbb{1}_{|y| \geq r^{-1}} d\mu_{\alpha,\kappa}(y) \Big\} \\ &\overset{r\to\infty}{\longrightarrow} \exp\Big\{ \int \left(e^{iuy} - 1\right) d\mu_{\alpha,\kappa}(y) \Big\} \end{aligned}$$

$$\begin{aligned} \mathbf{E}\left(|Z_{n,r}|\right) &\leq n a_n^{-1} \mathbf{E}\left(|\xi_1| \mathbb{1}_{|\xi_1| < a_n/r}\right) \\ &= n a_n^{-1} \int_0^{a_n/r} x d\mathbf{P}\left(|\xi_1| < x\right) \\ &= n a_n^{-1} \int_0^{a_n/r} \big\{ \mathbf{P}\left(|\xi_1| > x\right) - \mathbf{P}\left(|\xi_1| > a_n/r\right) \big\} dx \\ &= n a_n^{-1} \Big\{ \int_0^{a_n/r} x^{-\alpha} L(x) dx - (a_n/r)^{1-\alpha} L(a_n/r) \Big\} \\ &= n a_n^{-1} (a_n/r)^{1-\alpha} L(a_n/r) \left(\frac{\alpha}{1-\alpha} + o(1) \right) \\ &= \big\{ n a_n^{-\alpha} L(a_n) \big\} \frac{L(a_n/r)}{L(a_n)} \left(\frac{\alpha}{1-\alpha} + o(1) \right) r^{\alpha - 1}. \end{aligned}$$

Hence: $\lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{E}\left(|Z_{n,r}|\right) = 0$.

Case $\alpha \in (1,2)$

$$
\begin{aligned}
X_n := \frac{S_n - n\mathbf{E}(\xi_1)}{a_n} &= \sum_{j=1}^{n} \left\{ \frac{\xi_j}{a_n} - \mathbf{E}\left(\frac{\xi_j}{a_n}\right) \right\} \\
= \quad & \sum_{j=1}^{n} \left\{ \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} - \mathbf{E}\left(\frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} \right) \right\} \\
& + \sum_{j=1}^{n} \left\{ \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} < r^{-1}} - \mathbf{E}\left(\frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} < r^{-1}} \right) \right\} \\
=: & \; Y_{n,r} + Z_{n,r}.
\end{aligned}
$$

$$
\begin{aligned}
Y_{n,r} = \sum_{j=1}^{n} \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} & - n a_n^{-1} \mathbf{E}\left(\xi_1 1\!\!1_{\frac{\xi_1}{a_n} \geq r^{-1}} \right) \\
& - n a_n^{-1} \mathbf{E}\left(\xi_1 1\!\!1_{\frac{\xi_1}{a_n} \leq -r^{-1}} \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}\left(e^{iuY_{n,r}}\right) & \overset{n\to\infty}{\longrightarrow} \exp\left\{ \int \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{|y| \geq r^{-1}} d\mu_{\alpha,\kappa}(y) \right\} \\
& \overset{r\to\infty}{\longrightarrow} \exp\left\{ \int \left(e^{iuy} - 1 - iuy\right) d\mu_{\alpha,\kappa}(y) \right\}
\end{aligned}
$$

Computation below.

$$\begin{aligned}
&na_n^{-1}\mathbf{E}\left(\xi_1 1\!\!1_{\frac{\xi_1}{a_n}\geq r^{-1}}\right)\\
&\quad= na_n^{-1}\int_{a_n/r}^{\infty} x d\mathbf{P}\left(\xi_1<x\right)\\
&\quad= na_n^{-1}\int_{a_n/r}^{\infty}\big\{-d\big(x\mathbf{P}\left(\xi_1\geq x\right)\big)+\mathbf{P}\left(\xi_1\geq x\right)dx\big\}\\
&\quad= na_n^{-1}\big\{(a_n/r)^{1-\alpha}L^+(a_n/r)+\int_{a_n/r}^{\infty}x^{-\alpha}L^+(x)dx\big\}\\
&\quad= na_n^{-\alpha}L(a_n)\frac{L^+(a_n/r)}{L(a_n)}\big(\frac{\alpha}{\alpha-1}+o(1)\big)r^{\alpha-1}\\
&\quad\stackrel{n\to\infty}{\longrightarrow}\frac{\kappa+1}{2}\frac{\alpha}{\alpha-1}r^{\alpha-1}\\
&\quad= \frac{\kappa+1}{2}\int_{1/r}^{\infty}y\frac{\alpha}{y^{\alpha+1}}dy \qquad\qquad (\checkmark).
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}\left(Z_{n,r}^2\right) &\leq na_n^{-2}\big\{\mathbf{E}\left(\xi_1^2 1\!\!1_{|\xi_1|<a_n/r}\right)-\mathbf{E}\left(\xi_1 1\!\!1_{|\xi_1|<a_n/r}\right)^2\big\}\\
&= na_n^{-2}\big\{\int_0^{a_n/r}x^2 d\mathbf{P}\left(|\xi_1|<x\right)-\big(\int_0^{a_n/r}x d\mathbf{P}\left(|\xi_1|<x\right)\big)^2\big\}\\
&\leq na_n^{-2}\int_0^{a_n/r}x^2 d\mathbf{P}\left(|\xi_1|<x\right)\\
&= na_n^{-2}\big\{-(a_n/r)^2\mathbf{P}\left(|\xi_1|\geq a_n/r\right)+\int_0^{a_n/r}2x\mathbf{P}\left(|\xi_1|\geq x\right)dx\big\}\\
&= na_n^{-2}\big\{-(a_n/r)^{2-\alpha}L(a_n/r)+\left(\frac{2}{2-\alpha}+o(1)\right)(a_n/r)^{2-\alpha}L(a_n/r)\big\}\\
&= \big\{na_n^{-\alpha}L(a_n)\big\}\ \frac{L(a_n/r)}{L(a_n)}\ \left(\frac{\alpha}{2-\alpha}+o(1)\right)\ r^{\alpha-2}.
\end{aligned}$$

Hence: $\lim_{r\to\infty}\limsup_{n\to\infty}\mathbf{E}\left(Z_{n,r}^2\right)=0.$ □