Fizika I

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Tartalom

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Introduction - György Hárs

Present work is the summary of the lectures held by the author at Budapest University of Technology and Economics. Long verbal explanations are not involved in the text, only some hints which make the reader to recall the lecture. Refer here to the book: Alonso/Finn Fundamental University Physics, Volume I where more details can be found.

Physical quantities are product of a measuring number and the physical unit. In contrast to mathematics, the accuracy or in other words the precision is always a secondary parameter of each physical quantity. Accuracy is determined by the number of valuable digits of the measuring number. Because of this 1500 m and 1.5 km are not equivalent in terms of accuracy. They have 1 m and 100 m absolute errors respectively. The often used term relative error is the ratio of the absolute error over the nominal value. The smaller is the relative error the higher the accuracy of the measurement. When making operations with physical quantities, remember that the result may not be more accurate than the worst of the factors involved. For instance, when dividing 3.2165 m with 2.1 s to find the speed of some particle, the result 1.5316667 m/s is physically incorrect. Correctly it may contain only two valuable digits, just like the time data, so the correct result is 1.5 m/s.

The physical quantities are classified as fundamental quantities and derived quantities. The fundamental quantities and their units are defined by standard or in other words etalon. The etalons are stored in relevant institute in Paris. The fundamental quantities are the length, the time and the mass. The corresponding units are meter (m), second (s) and kilogram (kg) respectively. These three fundamental quantities are sufficient to build up the mechanics. The derived quantities are all other quantities which are the result of some kind of mathematical operations. To describe electric phenomena the fourth fundamental quantity has been introduced. This is ampere (A) the unit of electric current. This will be used extensively in Physics 2, when dealing with electricity.

1 Kinematics of a particle - György Hárs

Kinematics deals with the description of motion, without any respect to the cause of the motion. Strictly speaking there is no mass involved in the theory, so force and related quantities do not show up. The fundamental quantities involved are the length and the time only.

To describe the motion one needs a reference frame. Practically it is the Cartesian coordinate system with x, y, z coordinates, and corresponding i, j, k unit vectors.

The particle is a physical model. This is a point like mass, so it lacks of any extension.

1.1 Rectilinear motion

(Egyenes vonalú mozgás)

The motion of the particle takes place in a straight line in rectilinear motion. This means that the best mathematical description is one of the axes of the Cartesian coordinate system. So the position of the particle is described by $x(t)$ function.

The velocity of the particle is the first derivative of the position function. The everyday concept of speed is the absolute value of the velocity vector. Therefore the speed is always a nonnegative number, while the velocity can also be a negative number.

$$v(t) = \lim_{t=0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad \left[\frac{m}{s}\right] \tag{1.1}$$

The opposite direction operation recovers the position time function from the velocity vs. time function. Here x_0 is the initial value of the position in $t = 0$ moment, t' denotes the integration parameter from zero to t time.

$$x(t) = x_0 + \int_0^t v(t')dt' \tag{1.2}$$

The acceleration of the particle is the first derivative of the velocity vs. time function, thus it is the second derivative of the position vs. time function.

$$a(t) = \lim_{t=0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \left[\frac{m}{s^2}\right] \tag{1.3}$$

The opposite direction operation recovers the velocity time function from the acceleration vs. time function. Here v_0 is the initial value of the position in $t=$ 0 moment, t' denotes the integration parameter from zero to t time.

$$v(t) = v_0 + \int_0^t a(t')dt' \tag{1.4}$$

1.1.1 Uniform Rectilinear Motion

Here the acceleration of the particle is zero. The above formulas transform to the following special cases. $a=$ 0, v $= v_0, x = x_0+$ vt.

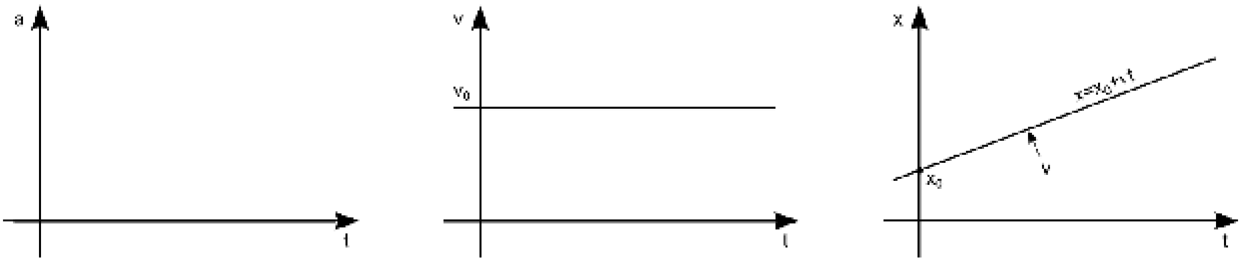


Figure 1: Uniform Rectilinear Motion

1.1.2 Uniformly Accelerated Rectilinear Motion

Here the acceleration of the particle is constant. The above formulas transform to the following special cases. a $=$ const, $v = v_0 + at$ $x = x_0 + v_0t + \frac{a}{2}t^2$

Typical example is the free fall, where the acceleration is a $=$ g $=$ 9.81 m/s^2.

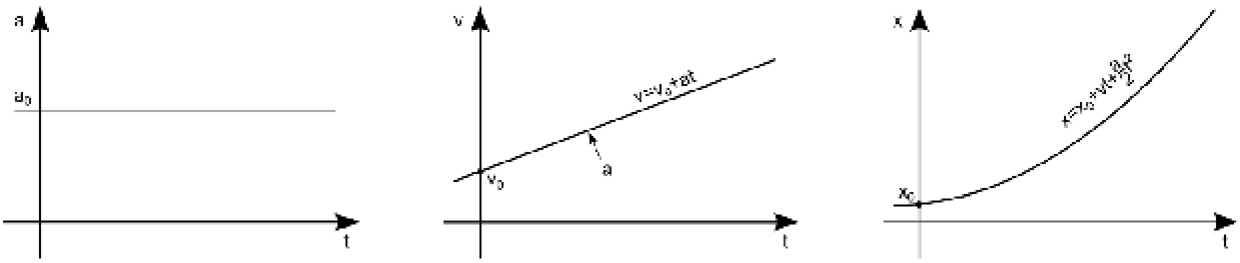


Figure 2: Uniformly Accelerated Rectilinear Motion

1.1.3 Harmonic oscillatory motion

The trajectory of the harmonic oscillation is straight line, so this is a special rectilinear motion. First let us consider a particle in uniform circular motion.

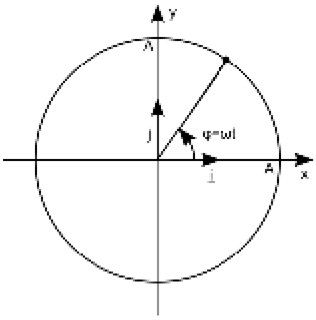


Figure 3: Harmonic oscillatory motion

The two coordinates in the Cartesian coordinate system are as follows:

$$x = A\cos(\omega t + \phi) \quad y = A\sin(\omega t + \phi) \tag{1.5}$$

If the uniform circular motion is projected to one of its coordinates, the motion of the projected point is "harmonic oscillatory motion". We choose the x coordinate.

$$x(t) = A\cos(\omega t + \phi) \tag{1.6}$$

The displacement at oscillatory motion is called excursion. The sum in the parenthesis is called the "phase". The multiplier of time is called angular frequency, and additive φ constant is the initial phase. The multiplier in front is called the "amplitude". The velocity of the oscillation is the derivative of the displacement function.

$$\frac{dx(t)}{dt} = v(t) = -A\omega\sin(\omega t + \phi) \tag{1.7}$$

The multiplier of the trigonometric term is called the "velocity amplitude" (v_{max}) .

$$v_{\max} = A\omega \tag{1.8}$$

The acceleration is the derivative of the velocity:

$$\frac{dv(t)}{dt} = a(t) = -A\omega^2 \cos(\omega t + \phi) \tag{1.9}$$

If one compares the displacement and the acceleration functions the relation below can readily found:

$$a(t) = -\omega^2 x(t) \tag{1.10}$$

Accordingly, the acceleration is always opposite phase position relative to the displacement.

In the kinematics of the harmonic oscillations it is very much helpful to go back to the origin of the oscillatory motion and contemplate the phenomena as projected component of a uniform circular motion. This way one gets rid of the trigonometric formalism and the original problem could have a far easier geometric interpretation. Best example for that if we want to find out the resultant oscillation of two identical frequency harmonic oscillations with different amplitudes and different initial phases. In pure trigonometry approach this is a tedious work, while in the circle diagram this is a simple geometry problem, actually a cosine theorem application in the most ordinary case.

1.2 Curvilinear motion

(Görbervonalú mozgás)

The motion of the particle is described by an arbitrary r(t) vector scalar function, where i, j, k are the unit vectors of the coordinate system.

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{1.11}$$

The velocity of the particle is the first derivative of the position function.

$$\mathbf{v}(t) = \lim_{t=0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{1.12}$$

The velocity vector is tangential to the trajectory of the particle always.

The vector of acceleration is the derivative of the velocity vector. The vector of acceleration can be decomposed as parallel and normal direction to the velocity.

$$\mathbf{a}(t) = \lim_{t=0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{1.13}$$

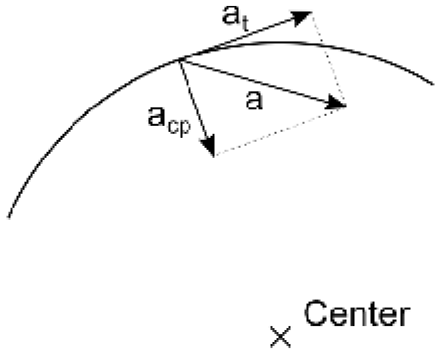


Figure 4: Curvilinear motion

The parallel component of the acceleration (called tangential acceleration) is the consequence of the variation in the absolute value of the velocity. In other words this is caused by the variation of the speed. The normal component of the acceleration (called centripetal acceleration) is the consequence of the change in the direction of the velocity vector.

If one drives a car on the road, speeding up or slowing down causes the tangential acceleration to be directed parallel or opposite with the velocity, respectively. By turning the steering wheel, centripetal acceleration will emerge. The direction of the centripetal acceleration points in the direction of the virtual center of the bend.

1.2.1 Projectile motion

(Hajítás)

In the model of the description the following conditions will be used:

Projectile is a particle,

Gravity field is homogeneous,

Rotation of the Earth, does not take part,

No drag due to air friction will be considered.

In real artillery situation the phenomenon is much more complex. This is far beyond the present scope.

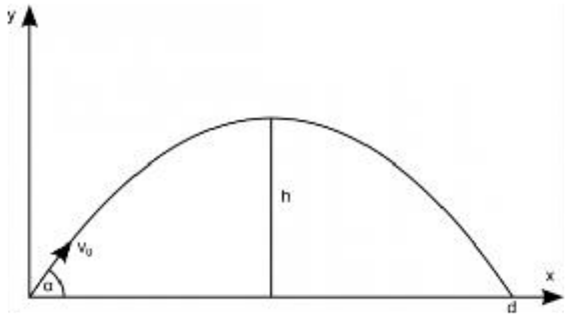


Figure 5: Projectile motion

The projectile is fired from the origin of the Cartesian coordinate system. The motion is characterized by the initial velocity v_0 and the angle of the velocity α relative to the horizontal direction. The motion will take place in the vertical plane, which contains the velocity vector. The motion is the superposition of a uniform horizontal rectilinear motion, a uniform vertical rectilinear motion and a free fall. Thus the velocity components are as follows:

$$v_x = v_0 \cos\alpha \tag{1.14}$$

$$v_y = v_0 \sin\alpha - gt \tag{1.15}$$

The corresponding position coordinates are the integrated formulas with zero initial condition.

$$x = \int_0^t v_x(t^{'})dt^{'} = v_0 t \cos\alpha \tag{1.16}$$

$$y = \int_0^t v_y(t^{'})dt^{'} = v_0 t \sin\alpha - \frac{g}{2}t^2 \tag{1.17}$$

Two critical parameters are needed to find out. These are the height of the trajectory (h) and the horizontal flight distance (d). First, the rise time should be calculated. The rise time τ_{rise} is the time when the vertical velocity component vanishes. Accordingly $v_y =$0 condition should be met. From the equation the following results:

$$\tau_{rise} = \frac{v_0 \sin\alpha}{g} \tag{1.18}$$

The height of the trajectory shows up as a vertical coordinate just in rise time moment.

$$h = y(t = \tau_{rise}) = v_0 \tau_{rise} \sin\alpha - \frac{g}{2}\tau_{rise}^2 \tag{1.19}$$

By substituting the formula of τ_{rise} into the equation above, the height of the trajectory results:

$$h = \frac{v_0^2 \sin^2\alpha}{g} - \frac{g}{2}\frac{v_0^2 \sin^2\alpha}{g^2} \tag{1.20}$$

Accordingly:

$$h = \frac{v_0^2 \sin^2\alpha}{2g} \tag{1.21}$$

The rise and the fall part of the motion last the same duration, due to the symmetry of the motion. Because of this, the total flight time of the motion is twice longer than the rise time alone. The horizontal flight distance (d) can be calculated as the horizontal x coordinate at double rise time moment.

$$d = x(t = 2\tau_{rise}) = v_0 2\frac{v_0 \sin\alpha}{g}\cos\alpha = \frac{v_0^2}{g} 2\sin\alpha\cos\alpha \tag{1.22}$$

By using elementary trigonometry, the final formula of horizontal flight distance results:

$$d = \frac{v_0^2}{2}\sin 2\alpha \tag{1.23}$$

This clearly shows that the projectile flies the furthest if the angle of the shot is 45 degrees.

1.2.2 Circular motion

In circular motion, the particle moves on a circular plane trajectory. To describe the position of the particle polar coordinates are used. The origin of the polar coordinate system is the center of the motion. The only variable parameter is the angular position $\varphi(t)$ since the radial position is constant.

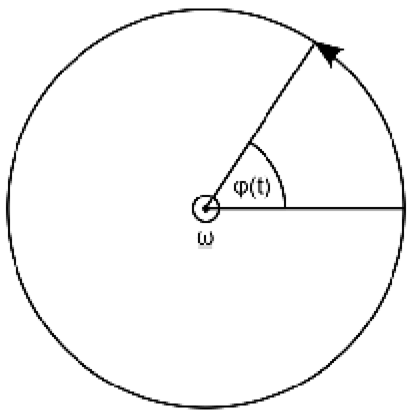


Figure 6: Circular motion

The derivative of the angular position is the angular velocity ω .

$$\omega(t) = \lim_{t=0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt} \quad \left[\frac{1}{s}\right] \tag{1.24}$$

Up to this moment it looks as if the angular velocity were a scalar number. But this is not the case. The angular velocity is a vector in fact, because it should contain the information about the rotational axis as well. By definition, the angular velocity vector ω is as follows: The absolute value of the ω is the derivative of the angular position as written above. The direction of the ω is perpendicular, or in other words, normal to the plane of the rotation, and the direction results as a right hand screw rotation. This latter means that by turning a usual right hand screw in the direction of the circular motion, the screw will proceed in the direction of the ω vector. Just an example: If the circular motion takes place in the plane of this paper and the rotation is going clockwise, the ω will be directed into the paper. Counter clockwise rotation will obviously result in a ω vector pointing upward, away from the paper.

With the help of ω vector number of calculation will be much easier to carry out. For example finding out the velocity vector of the particle is as easy as that:

$$\mathbf{v} = \omega \times \mathbf{r} \tag{1.25}$$

This velocity vector is sometimes called "circumferential velocity" however this notation is redundant, since the velocity vector is always tangential to the trajectory. The cross product of vectors in mathematics has a clear definition. By turning the first factor (ω) into the second one (r) the corresponding turning direction defines the direction of the velocity vector by the right hand screw rule. The absolute value of the velocity is the product of the individual absolute values, multiplied with the sine of the angle between the vectors.

Before going into further details, let us state three mathematical statements. Let a(t) and b(t) are two time dependent vectors and λ (t) a time dependent scalar. Then the following differentiation rules apply:

$$\frac{d}{dt}\left(\mathbf{a}(t)\times\mathbf{b}(t)\right)=\frac{d\mathbf{a}(t)}{dt}\times\mathbf{b}(t)+\mathbf{a}(t)\times\frac{d\mathbf{b}(t)}{dt} \tag{1.26}$$

$$\frac{d}{dt}\left(\mathbf{a}(t)\mathbf{b}(t)\right)=\frac{d\mathbf{a}(t)}{dt}\mathbf{b}(t)+\mathbf{a}(t)\frac{d\mathbf{b}(t)}{dt} \tag{1.27}$$

$$\frac{d}{dt}\left(\lambda(t)\mathbf{a}(t)\right)=\frac{d\lambda(t)}{dt}\mathbf{a}(t)+\lambda(t)\frac{d\mathbf{a}(t)}{dt} \tag{1.28}$$

These formulas make it possible to use the same differentiation rules among the vector products, just like among the ordinary product functions. End this is true both the cross product and the dot product operations. The proof of these rules, are quite straightforward. The vectors should be written by components, and the match of the two sides should be verified.

Using the ω vector is a powerful means. This way the acceleration vector of the particle can be determined with a relative ease.

$$\mathbf{a}=\frac{d\mathbf{v}(t)}{dt}=\frac{d}{dt}\left(\omega\times\mathbf{r}\right)=\frac{d\omega}{dt}\times\mathbf{r}+\omega\times\frac{d\mathbf{r}}{dt} \tag{1.29}$$

The derivative of ω vector is called the vector of angular acceleration β. This is the result of the variation in the angular velocity either due to spinning faster or slower or by changing the axis of the rotation.

$$\frac{d\omega}{dt}=\beta \quad \left[\frac{1}{s^2}\right] \tag{1.30}$$

Last term is the derivative of the position vector. This is the velocity, which can be written as above wit the help of ω vector. So ultimately the acceleration vector can be summarized.

$$\mathbf{a}=\beta\times\mathbf{r}+\omega\times\left(\omega\times\mathbf{r}\right) \tag{1.31}$$

The above formula consists of two major terms. The first term is called tangential acceleration. In case of plane motion, this is parallel or opposite to the velocity and it is the consequence of speeding up or slowing down, as explained in the earlier part of this chapter. The second term is called the centripetal or normal acceleration. This component points toward the center of the rotation. The centripetal acceleration is the consequence of the direction variation of the velocity vector. The absolute values of these components can readily be expressed.

$$a_{\tan} = \beta r \quad a_{cpt} = r\omega^2 = \frac{v^2}{r} \tag{1.32}$$

There are two special kinds of circular motion, the uniform and the uniformly accelerating circular motion.

1.2.2.1 Uniform circular motion:

In here the angular velocity is constant. The angle or rotation can be expressed accordingly:

$$\phi(t) = \omega t + \phi_0 \tag{1.33}$$

Since the angular acceleration is zero, no tangential acceleration will emerge. However there will be a constant magnitude centripetal acceleration, with an ever changing direction, pointing always to the center.

$$a_{cpt} = r\omega^2 = \frac{v^2}{r} \tag{1.34}$$

1.2.2.2 Uniformly accelerating circular motion

In here the angular acceleration is constant. The corresponding formulas are analogous to that of uniformly accelerating rectilinear motion, explained earlier in this chapter.

$$\omega(t) = \beta t + \omega_0 \tag{1.35}$$

$$\phi(t) = \frac{\beta}{2}t^2 + \omega_0 t + \phi_0 \tag{1.36}$$

The magnitude of the tangential acceleration is constant and parallel with the velocity vector.

$$a_{\tan} = \beta r = const \tag{1.37}$$

The magnitude of the centripetal component shows quadratic dependence in time.

$$a_{cpt} = r\omega^2 = r(\beta t + \omega_0)^2 \tag{1.38}$$

1.2.3 Areal velocity

(Területi sebesség)

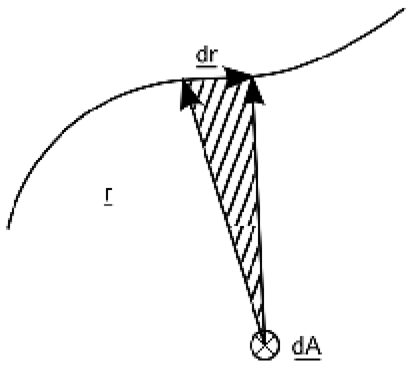


Figure 7: Areal velocity

Let us consider particle travelling on its trajectory. If one draws a line between the origin of the coordinate system and the particle, this line is called the "radius vector". The vector of areal velocity is the ratio of the area swept by the radius vector over time. The crosshatched triangle on the figure above is the absolute value of the infinitesimal variation (d A) of the swept area vector.

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r} \tag{1.39}$$

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times \mathbf{v}dt \tag{1.40}$$

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} \quad \left[\frac{m^2}{s}\right] \tag{1.41}$$

Areal velocity will be used in the study of planetary motion later in this book.

2 Dynamics of a Particle - György Hárs

(Tömegpont dinamikája)

Dynamics deals with the cause of motion. So in dynamics a new major quantity shows up. This is the mass of the particle (m). The concept of force and other related quantities will be treated as well. In this chapter only one piece of particle will be the subject of the discussion, in the next chapter however the system of particles will be treated.

2.1 Inertial system

In kinematics any kind of coordinate system could be used, there was no restriction in this respect. In dynamics however, a dedicated special coordinate system is used mostly. This is called inertial system. The inertial system is defined as a coordinate system in which the law of inertia is true. The law of inertia or Newton's first law says that the motion state of a free particle is constant. This means that if it was standstill it stayed standstill, if it was moving with a certain velocity vector, it continues its motion with the same velocity. So the major role of Newton's first law is the definition of the inertial system. Other Newton's laws use the inertial system as a frame of reference further on. The best approximation of the inertial system is a free falling coordinate system. In practice this can be a space craft orbiting the Earth, since the orbiting space craft is in constant free fall.

The inertial systems are local. This means that the point of the experimentation and its relative proximity belongs to a dedicated inertial system. An example explains this statement: Imagine that we are on a huge spacecraft circularly orbiting the Earth, so we are in inertial system. Now a small shuttle craft is ejected mechanically from the spacecraft without any rocket engine operation. The shuttle craft also orbits the Earth on a different trajectory and departs relatively far from the mother ship. Observing the events from the inertial system of the mother ship the shuttle supposed to keep its original ejection velocity and supposed to depart uniformly to the infinity. Much rather instead the shuttle craft also orbits the Earth and after a half circle it returns to the mother ship on its own. So the law of inertia is true in the close proximity of the experiment only. If one goes too far the law of inertia looses validity.

On the surface of the Earth we are not in inertial system. Partly because we experience weight, which is the gravity force attracting the objects toward the center, partly because the Earth is rotating, which rotation causes numerous other effects. Even though in most cases phenomena on the face of our planet can be described in inertial system, by ignoring the rotation related effects, and by considering the gravity a separate interaction.

2.2 The mass

Mass is a dual face quantity. Mass plays role in the interaction with the gravity field. This type of mass called gravitational mass and this is something like gravitational charge in the Newton's gravitational law.

$$F = G\frac{m_1 m_2}{r^2} \tag{2.1}$$

Here the m_1 and m_2 are the gravitational masses, r is the distance between the objects, F is the resulting force and G is the gravitational constant (6.67×10^{-11} m^3 kg^{-1} s^{-2}). When somebody measures the body weight with a bathroom scale he actually measures the gravitational mass.

Other major feature is that the mass shows resistance against the accelerating effect. This resistance is characterized by the inertial mass. It has been discovered later that these fundamentally different features can be related to the same origin, and so the two types of mass are equivalent. Therefore the distinction between them became unnecessary.

This equivalency makes the free falling objects drop with the same acceleration. The gravity force is proportional with the gravitational mass, which force should be equal with the acceleration times the inertial mass. So if the ratio of these masses were different, then the free fall would happen with different acceleration for different materials. This is harshly against the experience, so mass will be referred without any attribute later in this book.

2.3 Linear momentum p

(Impulzus, mozgásmennyiség, lendület)

By definition the linear momentum is the product of the mass and the velocity. Therefore linear momentum is a vector quantity.

$$\mathbf{p} = m\mathbf{v} \quad \left[\frac{kgm}{s}\right] \tag{2.2}$$

*Newton's second law:

This law is the definition of force (F).

The force exerted to a particle is equal to the time derivative of the linear momentum. The unit of force is Newton (N).

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \quad \left[\frac{kgm}{s^2}\right] = N \tag{2.3}$$

Conclusion 1.

If the force equals to zero, then the linear momentum is constant. This is in agreement with the law of inertia. However it is worth mentioning, that it is only true in inertial system. Which means that on an accelerating train or in a spinning centrifuge it is not valid.

Conclusion 2.

According to the fundamental theorem of calculus, the time integral of force results in the variation of the linear momentum:

$$\mathbf{p}_2 - \mathbf{p}_1 = \int_{t_1}^{t_2} \mathbf{F}(t)dt \tag{2.4}$$

The right hand side is called impulse (erőlökés).

Conclusion 3.

The well-known form of the Newton second law can be readily expressed:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{2.5}$$

Or briefly:

$$\mathbf{F} = m\mathbf{a} \tag{2.6}$$

*Newton third law:

(Action reaction principle)

When two particles interact, the force on one particle is equal value and opposite direction to the force of the other particle.

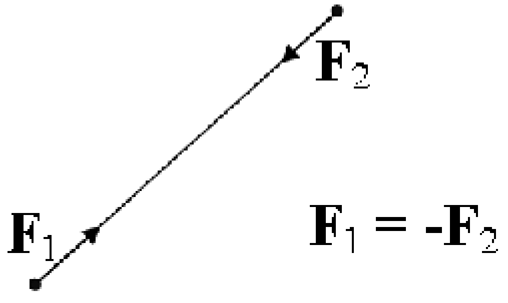


Figure 8: Action reaction principle

2.4 Equation of motion:

The particle is affected by numerous forces. The sum of these forces, cause the acceleration of the particle. This leads to a second order ordinary differential equation. This is called the equation of motion:

$$\sum_i \mathbf{F}_i(\mathbf{r}, t, \mathbf{v}) = m\frac{d^2\mathbf{r}}{dt^2} \tag{2.7}$$

In principle the forces may be the function of position, time and velocity.

*Example 1 for the equation of motion:

Attenuated oscillation:

(csillapodó rezgés)

A particle is hanging on a spring in water in vertical position. The particle is deflected to a higher position, and left alone to oscillate. Describe the motion by solving the equation of motion. Ignore the buoyant force. The motion will take place in the vertical line. The position is denoted $y(t)$ which is positive upside direction.

The forces affecting the particle are as follows:

$$F_{spring} = -Dy \quad F_{drag} = -k\frac{dy}{dt} \quad F_{grav} = -mg \tag{2.8}$$

Here D is the direction constant of the spring in N/m units, k is the drag coefficient and g is 9.81 m/s^2 .Accordingly the equation of motion can be written:

$$-k\frac{dy(t)}{dt} - Dy(t) - mg = m\frac{d^2y(t)}{dt^2} \tag{2.9}$$

Ordering it to the form of a differential equation:

$$\frac{d^2y(t)}{dt^2} + \frac{k}{m}\frac{dy(t)}{dt} + \frac{D}{m}y(t) = g \tag{2.10}$$

Let us introduce β for the attenuation coefficient with the following definition:

$$\beta = \frac{k}{2m} \tag{2.11}$$

$$\frac{d^2y(t)}{dt^2} + 2\beta\frac{dy(t)}{dt} + \frac{D}{m}y(t) = g \tag{2.12}$$

The mathematical method for solving this differential equation is beyond the scope of this chapter. The solution below can be verified by substitution:

$$y(t) = Ae^{-\beta t}\cos\omega t - \frac{mg}{D} \tag{2.13}$$

Here A is the original value of the deflection, ω_0 is called the Thomson angular frequency and ω is the angular frequency of the attenuated oscillation with the following definitions:

$$\omega_0 = \sqrt{\frac{D}{m}} \quad \omega = \sqrt{\omega_0^2 - \beta^2} \tag{2.14}$$

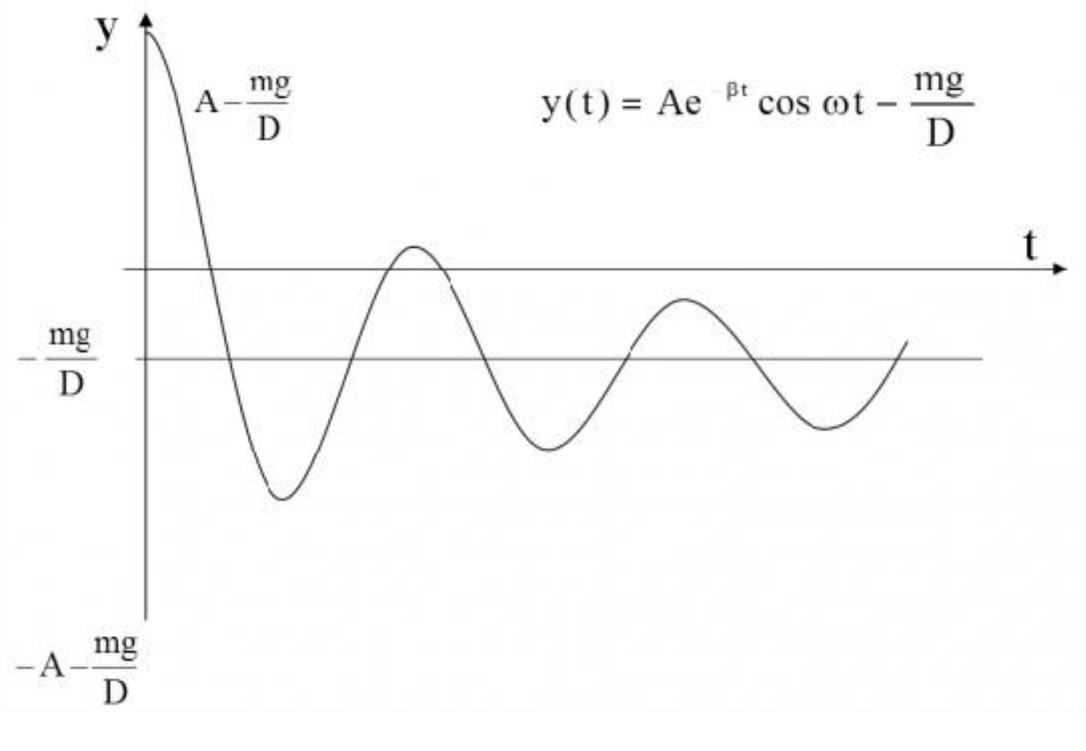


Figure 9:

*Example 2 for the equation of motion:

Conical pendulum

(kúpinga)

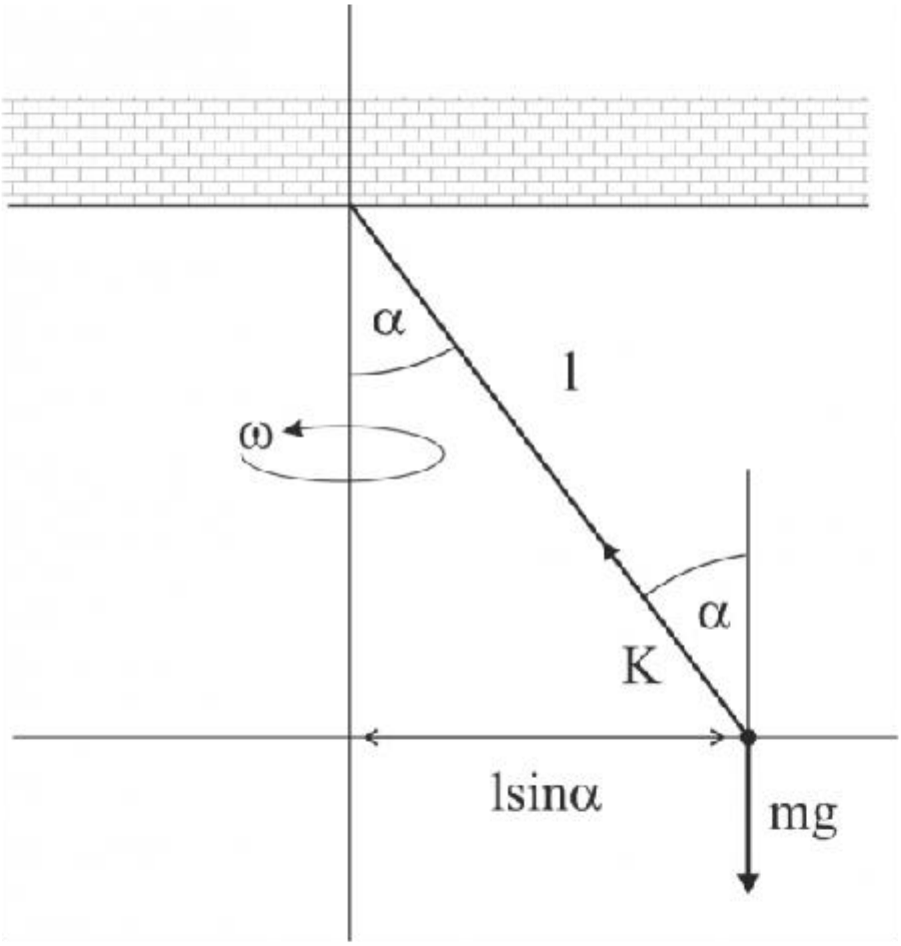


Figure 10: Conical pendulum

The conical pendulum circulates in horizontal plane with ω angular frequency. The angle of the rope α relative to the vertical direction is the unknown parameter to be determined. The coordinate system is an inertial system with horizontal and vertical axes, with the particle in the origin. There are two forces affecting the particle, gravity force (mg) and the tension of the rope (K). The equation of the motion is a vector equation in two dimensions so two scalar equations are used.

$$\leftarrow K \sin\alpha = m a_{cp} \tag{2.15}$$

$$\downarrow mg - K \cos\alpha = 0 \tag{2.16}$$

In addition the centripetal acceleration can be expressed readily:

$$a_{cp} = l\omega^2 \sin\alpha \tag{2.17}$$

After substitution $K = ml\omega^2$ results.

By means of this result the cosine of the angular position is determined:

$$\cos\alpha = \frac{g}{l\omega^2} \tag{2.18}$$

2.5 The concept of weight

Let us place a bathroom scale on the floor of an elevator. The normal force (N) is displayed by the scale that is transferred to the object.

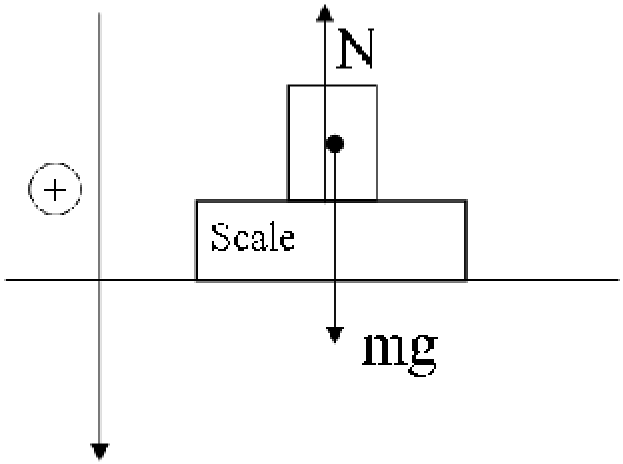


Figure 11: The concept of weight

The positive reference direction is pointing down. The following equation of motion can be written:

$$\downarrow mg - N = ma \tag{2.19}$$

Here the acceleration of the elevator is denoted (a) . Let us express the normal force indicated by the scale:

$$N = m(g - a) \tag{2.20}$$

If the elevator does not accelerate (in most cases it is standstill) the scale shows the force which is considered the weight of the object in general. ($N =$mg). This force is just enough to compensate the gravity force, so the object does not accelerate. However, when the elevator accelerates up or down, the indicated value is increased or decreased, respectively. This also explains that in a freefalling coordinate system, where $a = g$, the weight vanishes. Similarly zero gravity shows up on the orbiting spacecraft, which is also in constant freefall.

2.6 The concept of work in physics

The concept of work in general is very broad. Besides physics, it is used in economy, also used as "spiritual work". Concerning the physical concept, the amount of work is not too much related, how much tiredness is suffered by the person who actually made this work. For example, if somebody is standing with fifty kilogram sack on his back for an hour without any motion, surely becomes very tired. Furthermore if this person walks on a horizontal surface during this time, he gets tired even more. Physical work has not been done in either case.

In high school the following definition was learnt. "The work equals the product of force and the projected displacement". This is obviously true, but only for homogeneous force field and straight finite displacement. In equation: $\Delta W = \mathbf{F}\Delta\mathbf{r}$. Here we used the mathematical concept of dot product, which results in a scalar number, and the product of the two absolute values is multiplied with the cosine of the angle.

In general case when the related force field F(r) is not homogeneous and the displacement is not straight, the above finite concept is not applicable. We have to introduce the infinitesimal contribution of work (dW= F(r)d r). The amount of work made between two positions is the sum or in other words integral of dW contributions. The physical unit is Newton meter (Nm) which is called Joule (J) .

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}) d\mathbf{r} \quad \left[J = Nm = \frac{kgm^2}{s^2} \right] \tag{2.21}$$

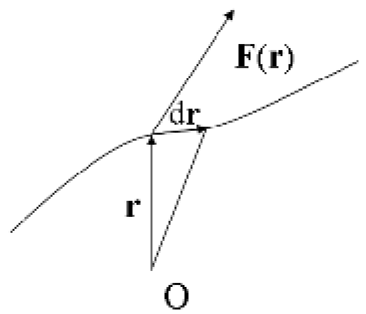


Figure 12: The concept of work

There is a special case when the force is the function of one variable $F(x)$ only, and its direction is parallel with the x direction. The above definition simplifies to the following:

$$W = \int_{x_1}^{x_2} F(x)dx \quad \left[J = Nm = \frac{kgm^2}{s^2}\right] \tag{2.22}$$

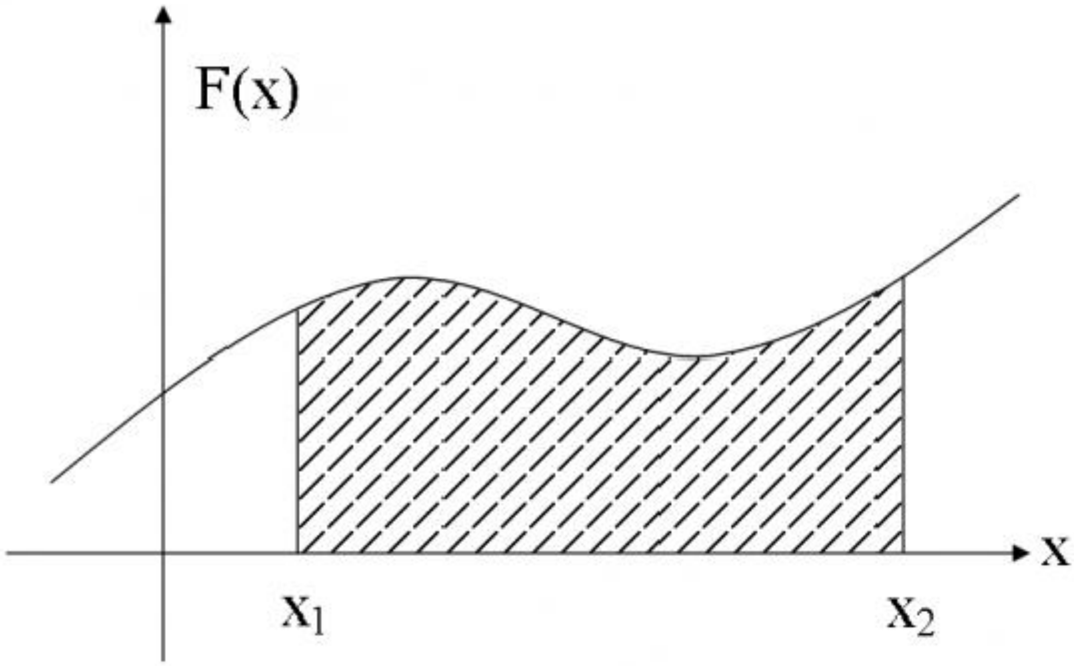


Figure 13: The work is the area under the curve

In this special case the work done between two positions is displayed by the area under the $F(x)$ curve.

2.7 Power

(Teljesítmény)

The power (P) is associated with the time needed to carry out a certain amount of work. In mathematics, this is the time derivative of the work done. The physical unit is Joule per second which is called Watt (W) .

$$P = \frac{dW}{dt} = \frac{d}{dt}(\mathbf{F}d\mathbf{r}) \quad \left[W = \frac{kgm^2}{s^3}\right] \tag{2.23}$$

Provided the force does not depend directly on time, the above formula can be transformed:

$$P = \frac{dW}{dt} = \frac{d}{dt}(\mathbf{F}d\mathbf{r}) = \mathbf{F}\mathbf{v} \tag{2.24}$$

So the instantaneous power is the dot product of the force and the actual velocity vector.

2.8 Theorem of Work (Kinetic energy)

Munkatétel (Mozgási energia)

Kinetic energy is the kind of energy which is associated with the mechanical motion of some object. In high school the following simplified argument was presented to calculate it:

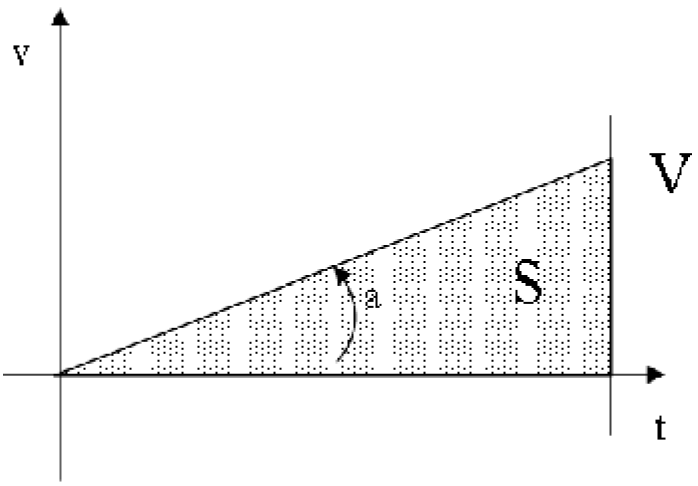


Figure 14: Velocity vs. time function by the effect of constant force

A particle with mass (m) is affected by constant force. Initially the particle is standstill. The acceleration is constant, thus the $v(t)$ graph is a sloppy line through the origin. After (t) time passed, the displacement (s) shows up as the area under the $v(t)$ curve. Its shape is a right angle triangle.

$$s = \frac{vt}{2} \tag{2.25}$$

The acceleration is the slope of the v(t) line.

$$a = \frac{v}{t} \tag{2.26}$$

Let us multiply the above equation with the mass of the particle:

$$ma = \frac{mv}{t} \tag{2.27}$$

The left hand side equals the force affecting the particle.

$$F = \frac{mv}{t} \tag{2.28}$$

We also know that the work done in this simple case is:

$$W = Fs \tag{2.29}$$

So let us substitute the related formulas. Time cancels out:

$$W = \frac{mv}{t}\frac{vt}{2} = \frac{1}{2}mv^2 \tag{2.30}$$

This is the work done on the particle which generated the kinetic energy.

The above argument is not general enough, due to the simplified conditions used. The general argument is presented below:

Let us start with Newton's second law:

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} \tag{2.31}$$

The work done in general is as follows:

$$W = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} d\mathbf{r} \tag{2.32}$$

Substitute first to the second formula:

$$W = \int_{\mathbf{r_1}}^{\mathbf{r_2}} m \frac{d^2\mathbf{r}}{dt^2} d\mathbf{r} \tag{2.33}$$

Switch the limits of the integration to the related time moments t_1 and t_2.

$$W = m \int_{t_1}^{t_2} (\frac{d^2\mathbf{r}}{dt^2} \frac{d\mathbf{r}}{dt}) dt \tag{2.34}$$

Take a closer look at the formulas in the parenthesis. In here the product of the first and the second derivative of some function are present.

The following rule is known in mathematics:

$$\frac{d}{dx} \left(\frac{1}{2} f^2(x) \right) = f(x) \frac{df(x)}{dx} \tag{2.35}$$

Using this formula for the last expression of work:

$$W = m \int_{t_1}^{t_2} \frac{d}{dt} \left[\frac{1}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 \right] dt = m \int_{t_1}^{t_2} d \left[\frac{1}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 \right] = \frac{m}{2} \int_{t_1}^{t_2} d \left[\mathbf{v}^2 \right] \tag{2.36}$$

By integrating the variations of the v^2, the total variation will be the result:

$$\frac{m}{2} \int_{t_1}^{t_2} d \left[\mathbf{v}^2 \right] = \frac{1}{2} m \mathbf{v}_2^2 - \frac{1}{2} m \mathbf{v}_1^2 \tag{2.37}$$

$$W = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} d\mathbf{r} = \frac{1}{2} m \mathbf{v}_2^2 - \frac{1}{2} m \mathbf{v}_1^2 = E_{kin2} - E_{kin1} \tag{2.38}$$

Thus: The work done on a particle equals the variation of the kinetic energy. This is the theorem of work.

Note there is no any restriction to the kind of force. So the force is not required to be conservative, which concept will be presented later in this chapter. This can be even sliding friction, drag or whatever other type of force.

The kinetic energy is accordingly:

$$E_{kin} = \frac{1}{2}m\mathbf{v}^2 \tag{2.39}$$

2.9 Potential energy

(Helyzeti energia)

Potential energy is the kind of energy which is associated with the position of some object in a force field. Force field is a vector-vector function in which the force vector F depends on the position vector r. In terms of mathematics the force field F(r) is described as follows:

$$\mathbf{F}(\mathbf{r}) = X(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + X(x,y,z)\mathbf{k} \tag{2.40}$$

where i, j, k are the unit vectors of the coordinate system.

Take a particle and move it slowly in the F(r) force field from position 1 to position 2 on two alternative paths.

Let us calculate the amount of work done on each path. The force exerted to the particle by my hand is just opposite of the force field -F(r). If it was not the case, the particle would accelerate. The moving is thought to happen quasi-statically without acceleration.

Let us calculate my work for the two alternate paths:

$$W_1 = \left(\int_{\mathbf{r_1}}^{\mathbf{r_2}} (-\mathbf{F})d\mathbf{r}\right)_{path1} \quad W_2 = \left(\int_{\mathbf{r_1}}^{\mathbf{r_2}} (-\mathbf{F})d\mathbf{r}\right)_{path2} \tag{2.41}$$

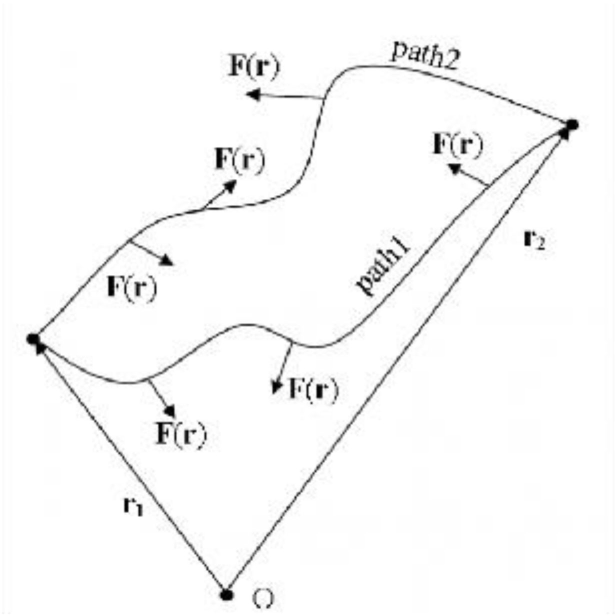


Figure 15: Integration on two alternative paths

In general case W_1 and W_2 are not equal. However, in some special cases they may be equal for any two paths. Imagine that our force field is such, that W_1 and W_2 are equal. In this case a closed loop path can be made which starts with path 1 and returns to the starting point on path 2. Since the opposite direction passage turns W_2 to its negative, ultimately the closed loop path will result in zero value. That special force field where the integral is zero for any closed loop is considered CONSERVATIVE force field. In formula:

$$\oint \mathbf{F}(\mathbf{r})d\mathbf{r} = 0 \tag{2.42}$$

At conservative force field, one has to choose a reference point. All other destination points can be characterized with the amount of the work done against the force field to reach the destination point. This work is considered the potential energy (E_{pot}) of the point relative to the reference point:

$$E_{pot}(\mathbf{r}) = \int_{\mathbf{ref}}^{\mathbf{r}} (-\mathbf{F}(\mathbf{r}'))d\mathbf{r}' = -\int_{\mathbf{ref}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')d\mathbf{r}' \tag{2.43}$$

The reference point can be chosen arbitrarily, however it is worth considering the practical aspects of the problem.

Due to the fact that the reference point is arbitrary, the value of the potential energy is also indefinite since direct physical meaning can only be associated to the variation of the potential energy. In other words, the individual potential energy values of any two points can be altered by changing the reference point, but the difference of the potential energy values does not change.

Now the work done against the forces of the field between $\mathbf{r_1}$ and $\mathbf{r_2}$ points can be expressed:

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} = \int_{\mathbf{r_1}}^{\mathbf{ref}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} + \int_{\mathbf{ref}}^{\mathbf{r_2}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} = -\int_{\mathbf{ref}}^{\mathbf{r_1}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} + \int_{\mathbf{ref}}^{\mathbf{r_2}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} \quad (2.44)$$

The last two integrals are the potential energies of $\mathbf{r_1}$ and $\mathbf{r_2}$ points respectively.

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} = E_{pot}(\mathbf{r_2}) - E_{pot}(\mathbf{r_1}) \quad (2.45)$$

2.10 Conservation of the mechanical energy

(Mechanikai energia megmaradása)

Mechanical energy consists of kinetic and potential energy by definition. Earlier in this chapter the theorem of work was stated. Work done on a particle equals the variation of its kinetic energy. In addition F(r) could be any kind of force.

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F}(\mathbf{r})d\mathbf{r} = E_{kin2} - E_{kin1} \quad (2.46)$$

Later the potential energy has been treated.

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} (-\mathbf{F}(\mathbf{r}))d\mathbf{r} = E_{pot}(\mathbf{r_2}) - E_{pot}(\mathbf{r_1}) \quad (2.47)$$

Let us switch the sign of the above equation:

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F}(\mathbf{r})d\mathbf{r} = E_{pot}(\mathbf{r_1}) - E_{pot}(\mathbf{r_2}) \quad (2.48)$$

At potential energy however conservative force field is required. This means that the so called dissipative interactions are excluded, such as the sliding friction and the drag. Let us make the right hand sides of the relevant equations equal.

$$E_{pot}(\mathbf{r_1}) - E_{pot}(\mathbf{r_2}) = E_{kin2} - E_{kin1} \tag{2.49}$$

Ordering the equation:

$$E_{pot}(\mathbf{r_1}) + E_{kin1} = E_{pot}(\mathbf{r_2}) + E_{kin2} \tag{2.50}$$

Using the conservation of mechanical energy requires conservative force, because this is the more stringent condition.

Ultimately let us declare again clearly the conservation of mechanical energy: In conservative system the sum of the kinetic and potential energy is constant in time. Accordingly, these two types of energy transform to each other during the motion, but the overall value is unchanged. In contrast to this when dissipative interaction emerges in the system, the total mechanical energy gradually decreases by heat loss.

In this chapter the concept of work end energy have been used extensively. To improve clarity, the following statement needs to be declared: Work is associated to some kind of process or action. Energy on the other hand is associated to some kind of state of a system, when not necessarily happens anything, but the capacitance to generate action is present.

2.11 Energy relations at harmonic oscillatory motion

The equation of motion of the harmonic oscillation is as follows:

$$F(t) = -Dx(t) \tag{2.51}$$

Here D is direction coefficient of the spring on which a particle with mass m oscillates.

In the chapter of kinematics the harmonic oscillatory motion has been introduced, and the basic formulae have all been derived. The following relation was recovered:

$$a(t) = -\omega^2 x(t) \tag{2.52}$$

Let us multiply it with mass:

$$ma(t) = -m\omega^2 x(t) \tag{2.53}$$

The left hand side of the equation is the force affecting the particle.

$$F(t) = -m\omega^2 x(t) \tag{2.54}$$

By comparing the two expressions of the force one can conclude as follows:

$D = m\omega^2$		$\omega = \sqrt{\frac{D}{m}}$

The harmonic oscillatory motion is a conservative process. This means that the total mechanical energy (the sum of kinetic and the potential energy) should be constant.

$$E_{tot} = \frac{1}{2}Dx^2 + \frac{1}{2}mv^2 = const \tag{2.55}$$

Let us verify the above statement with the concrete formulas of displacement and velocity:

$$E_{tot} = \frac{1}{2}DA^2\cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = const \tag{2.56}$$

Now we can proceed on two alternate tracks by substituting the direction coefficient into the equation and using the most basic trigonometric relation:

$$E_{tot} = \frac{1}{2}DA^2\cos^2(\omega t + \phi) + \frac{1}{2}DA^2 \sin^2(\omega t + \phi) = \frac{1}{2}DA^2 \tag{2.57}$$

Or alternatively:

$$E_{tot} = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}m\omega^2 A^2 \tag{2.58}$$

By using the velocity amplitude (v_{max}) defined in the chapter of kinematics one can conclude as follows:

$$E_{tot} = \frac{1}{2}mv_{\max}^2 \tag{2.59}$$

Ultimately we found two alternate formulae for the total mechanical energy. These formulae prove that the process is truly conservative, and the total energy may show up either as potential or kinetic energy. In amplitude position the total energy is stored in the spring as potential (elastic) energy, at zero excursion position the total energy is kinetic energy.

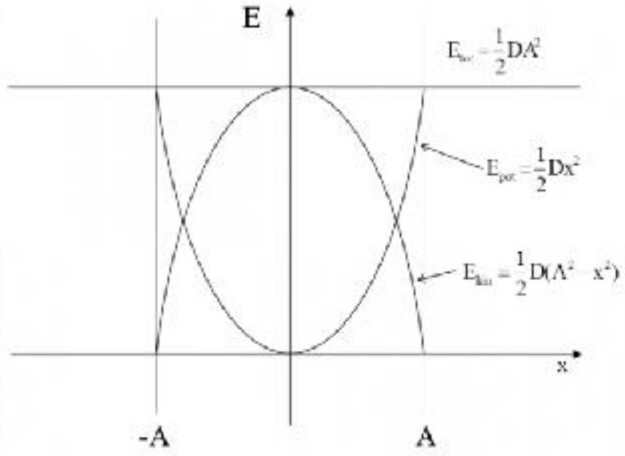


Figure 16: Energy relations of the oscillatory motion

In the figure above the energy relations are displayed. The motion takes place under the solid horizontal line of total energy.

2.12 Angular momentum

(Impulzus nyomaték, perdület)

By definition the angular momentum of the particle is the cross product of the position vector and the linear momentum.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad \left[\frac{kgm^2}{s}\right] \tag{2.60}$$

2.13 Torque

(Forgató nyomaték)

By definition the torque (M) is the cross product of the position vector and the force affecting the particle.

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad \left[\frac{kgm^2}{s^2}\right] \tag{2.61}$$

Let us consider the situation when r and p and F are in the plane of the sheet. According to the definition, both the angular momentum and the torque are normal to the sheet.

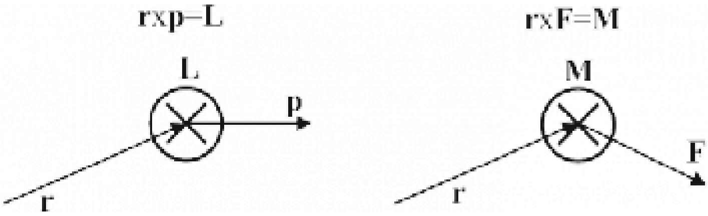


Figure 17: Both the angular momentum and the torque point into the paper

If the vectors depend on time, one can determine the derivative of the product:

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{2.62}$$

Since $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ and $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ the above equation can be transformed:

$$\frac{d\mathbf{L}}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} \tag{2.63}$$

The first term on the right cancels out because v and p vectors are parallel. Therefore:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} \tag{2.64}$$

The product on the right hand side is the torque. Ultimately one can conclude:

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} \tag{2.65}$$

In words: The time derivative of the angular momentum of some particle equals the torque affecting this particle. (Obviously the reference point of both L and M must be the same.)

This formula is analogous to that of Newton's second law, expressed with the linear momentum. By means of the fundamental theorem of calculus, this formula can be integrated.

$$\mathbf{L}_2 - \mathbf{L}_1 = \int_{t_1}^{t_2} \mathbf{M}(t)dt \tag{2.66}$$

In words:

The variation of the angular momentums is the time integral of the torque affecting the particle. This integral is called the angular impulse. (Nyomaték lökés)

2.14 Central force field

(Centrális erőtér)

If the force is collinear with the position vector and the magnitude depends on the distance alone, then the force field is considered central force field:

$$\mathbf{F} = k(r)\mathbf{r} \tag{2.67}$$

here k is a scalar number which may depend only on the distance from the center.

As it has already been calculated:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} \tag{2.68}$$

Let us substitute the central force field:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times k(r)\mathbf{r} \tag{2.69}$$

The cross product is zero because of the collinear arrangement:

$$\frac{d\mathbf{L}}{dt} = 0 \tag{2.70}$$

Accordingly, in central force field the angular momentum is constant in time: (L $=$ const) It has the important conclusion. Planets, moons or spacecrafts which orbit their central body in the space also move in the central force field of gravity. Therefore the angular momentum referred to the central body is constant.

In the chapter of kinematics the concept of areal velocity was introduced in general. Accordingly:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} \tag{2.71}$$

On the other hand, the angular velocity is:

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} \tag{2.72}$$

By combining these two last equations:

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2m}\mathbf{L} = const \tag{2.73}$$

Ultimately the areal velocity is constant in the central force field.

Planetary motion: A meteorite is orbiting the sun on an ellipse trajectory. The ellipse trajectory is the consequence of the Newton's gravitational law. The constant areal velocity will make the meteorite travel faster when close to the sun and slower when it is far away. The crosshatched areas in the figure below are equal. So, the motion is far not uniform.

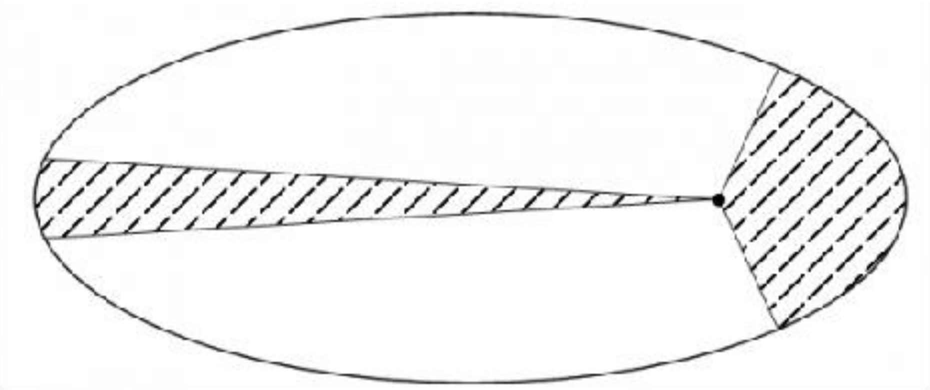
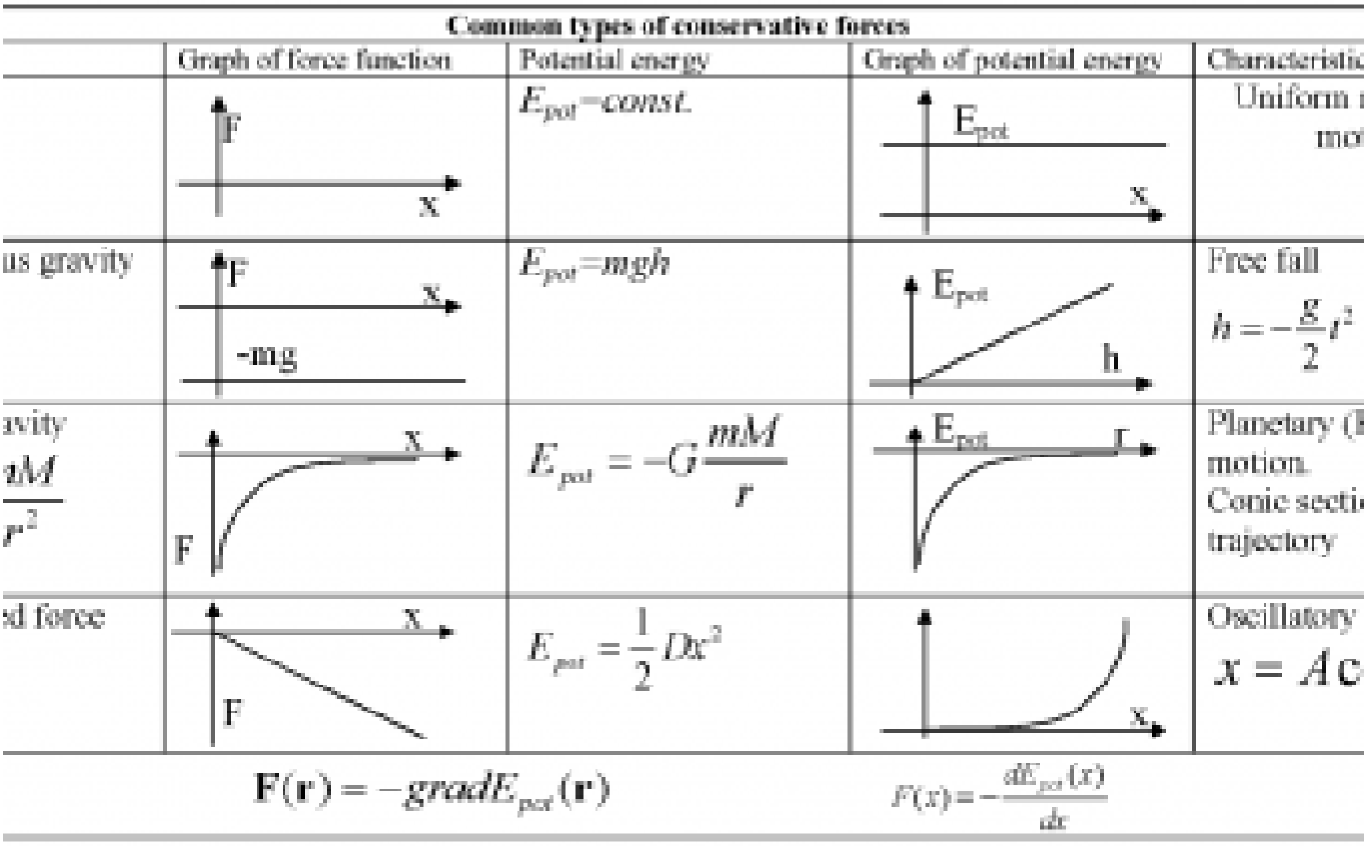


Figure 18: Planetary motion with constant areal velocity



Common types of conservative forces

	Graph of force function	Potential energy	Graph of potential energy	Characteristic
		$E_{pot}=const.$		Uniform mo
us gravity		$E_{pot}=mgh$		Free fall $h=-\frac{g}{2}t^2$
avity $\frac{mM}{r^2}$		$E_{pot}=-G\frac{mM}{r}$		Planetary (motion. Conic secti trajectory
d force		$E_{pot}=\frac{1}{2}Dx^2$		Oscillatory $x = Ac$
	$\mathbf{F}(\mathbf{r})=-gradE_{pot}(\mathbf{r})$		$F(x)=-\frac{dE_{pot}(x)}{dx}$	

Common types of friction related forces

	Force function	Power	Conservative or dissipative	Effect to
	$F_s \le \mu_s N$ N is normal force	0	Conservative (Non energetic)	v=0 in point
h	$F_d = \mu_d N$ $\mu_d \le \mu_s$	$P_d = -F_d v$ v is the speed	Dissipative	Point of con Speed slows
	$F_{drag} = -K\eta v$ K is form factor, η is viscosity	$P_{drag} = -K\eta v^2$	Dissipative	Point of con Speed slows

3 Dynamics of system of particles - György Hárs

(Tömegpont rendszer dinamikája)

3.1 Momentum in system of particles

The subject of analysis will be the system of particles. The system of particles in practice may consist of several particles (mass points). Each of the particles may travel arbitrarily in 3D space. The particles may exert force to each other (internal force) and may be affected by forces originating in the environment (external force).

In mathematical calculations however it is worth reducing the number of particles to two particles. This way, calculations become much easier without loosing generality. The physical meaning behind the equations becomes even more apparent. At the end of the argument the result will be stated in full generality for any number of particles.

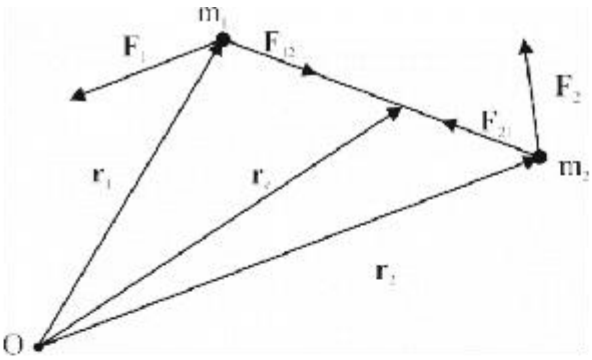


Figure 20: System of particles

The center of mass is the weighted average of the position vectors.

$$\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \mathbf{r_c} \tag{3.1}$$

Its time derivative is the velocity of the center of mass.

$$\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \mathbf{v_c} \quad \mathbf{p}_{tot} = \sum_i \mathbf{p}_i = \mathbf{v_c}\sum_i m_i \tag{3.2}$$

The numerator is the total momentum of the system of particles. So the total momentum can be expressed as the product of the velocity of the center of mass multiplied by the total mass.

Let us make one more time derivation:

$$\frac{m_1\mathbf{a_1} + m_2\mathbf{a_2}}{m_1 + m_2} = \mathbf{a_c} \tag{3.3}$$

Accordingly:

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = \mathbf{a_c}\sum_i m_i \tag{3.4}$$

Now consider the Newton equation for m_1 and m_2:

$$m_1\mathbf{a}_1 = \mathbf{F_1} + \mathbf{F_{12}} \quad m_2\mathbf{a}_2 = \mathbf{F}_2 + \mathbf{F_{21}} \tag{3.5}$$

Internal forces show up with double subscript. By substituting the forces to the above equation:

$$(\mathbf{F_1} + \mathbf{F_{12}}) + (\mathbf{F}_2 + \mathbf{F_{21}}) = \mathbf{a_c}\sum_i m_i \tag{3.6}$$

Here we have to take into account the fact, that the internal forces show up in pairs and they are opposite of each other. $F_{12} = -F_{21}$. So they cancel out and only the external forces remain.

$$\sum_i \mathbf{F}_{iext} = \mathbf{a_c}\sum_i m_i \tag{3.7}$$

In words: The sum of the external forces accelerates the center of mass. Internal forces do not affect the acceleration of the center of mass. This is the theorem of momentum.

If on the other hand the sum of the external forces is zero, the acceleration of the center of mass becomes also zero, or in other words, the velocity of the center of mass is constant. If the velocity of the center of mass is constant, then the total momentum of the system of particles will also be constant.

So all together, let us state the conservation of momentum: In an isolated mechanical system (in here the sum of the external forces is zero) the total momentum of the system of particles is constant.

$$\mathbf{p}_{tot} = Const \tag{3.8}$$

This law can also be used in coordinate components. So if the system of particles is mounted on a little rail cart, and external force parallel with the rail does not affect the system, then that component of the total momentum will be constant which is parallel with the rail. In terms of other directions no any law applies.

3.1.1 Collisions

(Ütközések)

At commonly happening collisions the conservation of momentum is valid because the system of the two colliding particles represents an isolated mechanical system. There are two specific types of collisions, the inelastic and elastic collision. The distinction is based on the kinetic energy variation during the process.

3.1.1.1 Inelastic collisions:

(Rugalmatlan ütközés)

The two colliding particles get stuck together. The kinetic energy of the system is partly dissipated. Substantial amount of heat can be generated. Let us write the conservation of momentum:

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{u} \tag{3.9}$$

The velocity after collision (u) results:

$$\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \mathbf{u} \tag{3.10}$$

The "lost" mechanical energy, which has been dissipated to heat, is the difference of the total kinetic energy before and after the collision:

$$\Delta E_{loss} = \frac{1}{2}m\mathbf{v}_1^2 + \frac{1}{2}m\mathbf{v}_2^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}\right)^2 \tag{3.11}$$

3.1.1.2 Elastic collision:

(Rugalmas ütközés)

Word "elastic" means that the mechanical energy is conserved. Thus, both the momentum and the kinetic energy are conserved. After collision the particles get separated with different velocities. The velocities before and after the collision are denoted with v_1 v_2 and u_1 u_2 respectively. The conservation of momentum follows:

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{u}_1 + m_2\mathbf{u}_2 \tag{3.12}$$

The conservation of mechanical energy is also valid. Here the total mechanical energy is kinetic energy since no potential energy is involved.

$$\frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_1\mathbf{v}_2^2 = \frac{1}{2}m_1\mathbf{u}_1^2 + \frac{1}{2}m_1\mathbf{u}_2^2 \tag{3.13}$$

Let us group terms with subscript 1 to the left and terms with subscript 2 to the right hand side for two equations above.

$$m_1\mathbf{v}_1 - m_1\mathbf{u}_1 = m_2\mathbf{u}_2 - m_2\mathbf{v}_2 \tag{3.14}$$

$$\frac{1}{2}m_1\mathbf{v}_1^2 - \frac{1}{2}m_1\mathbf{u}_1^2 = \frac{1}{2}m_2\mathbf{u}_2^2 - \frac{1}{2}m_2\mathbf{v}_2^2 \tag{3.15}$$

Now factor out m_1 and m_2 from the equations, multiply the kinetic energy equation with two and use the equivalency for the difference of squares:

$$m_1(\mathbf{v}_1 - \mathbf{u}_1) = m_2(\mathbf{u}_2 - \mathbf{v}_2) \tag{3.16}$$

$$m_1(\mathbf{v}_1 - \mathbf{u}_1)(\mathbf{v}_1 + \mathbf{u}_1) = m_2(\mathbf{u}_2 - \mathbf{v}_2)(\mathbf{u}_2 + \mathbf{v}_2) \tag{3.17}$$

Up to this point of discussion the 3D vector equations above are fully valid. Among dot products, division operation is impossible. This is due to the fact that reverse direction of the operation is ambiguous.

From this point, the mathematical argument is confined to the central collision only. At central collision the velocities before the collision are parallel with the line between the centers of the particle. This way the collision process takes place in a single line, and the velocities before and after the collision will all be 1D vectors in the line of the collision. The 1D vectors are practically plus, minus or zero numbers, and the dot product between these vectors is basically product between real numbers. So from the above equations the vector notation will be omitted. Accordingly any division can readily be carried out.

$$m_1(v_1 - u_1) = m_2(u_2 - v_2) \tag{3.18}$$

$$m_1(v_1 - u_1)(v_1 + u_1) = m_2(u_2 - v_2)(u_2 + v_2) \tag{3.19}$$

Let us divide the last equation with the former one:

$$v_1 + u_1 = v_2 + u_2 \tag{3.20}$$

Now group the v terms to the left and u terms to the right hand side:

$$v_1 - v_2 = u_2 - u_1 \tag{3.21}$$

Multiply the equation with m_2:

$$m_2 v_1 - m_2 v_2 = m_2 u_2 - m_2 u_1 \tag{3.22}$$

The original equation for momentum conservation is simplified for 1D central collision:

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \tag{3.23}$$

Let us subtract the former equitation from the last one. Here $m_2 u_2$ term will cancel out:

$$v_1(m_1 - m_2) + 2m_2 v_2 = u_1(m_1 + m_2) \tag{3.24}$$

Thus u_1 can be expressed:

$$v_1 \frac{m_1 - m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2} v_2 = u_1 \tag{3.25}$$

Due to symmetry, formula for u_2 can be easily derived by switching the subscripts 1 and 2.

$$v_2 \frac{m_2 - m_1}{m_1 + m_2} + \frac{2m_1}{m_1 + m_2} v_1 = u_2 \tag{3.26}$$

The above formulas are not simple enough to provide plausible results. For this purpose some special cases will be treated separately:

*Discussion 1:

What if $m_1 = m_2 = m$ is the case.

Basically the masses are equal. Then the final result simplifies to:

$v_2 = u_1$ and $v_1 = u_2$

Accordingly the particles swap their velocities. If on the other hand one of the particle had zero velocity originally $(v_1 =$ 0), and the other particle slammed into it with v_2 velocity. Then :

$v_2 = u_1$ and $u_2 = 0$

This means that the standing particle will start travelling with the velocity of the moving particle, and the originally moving particle will stop.

*Discussion 2:

What if m_2 is far larger than m_1.

$-v_1 + v_2 = u_1$ and $v_2 = u_2$

This is the case when a ball bounces back from the face of the incoming bus. The velocity of bus does not change ($v_2 = u_2)$, and the velocity of ball is reflected plus the speed of the buss is added.

*Discussion 3:

Billiard ball collision:

In this game, the balls are equal in mass, but the collisions are not necessarily central. Consider the situation when one ball is standing and an equal weight ball collides to it in a skew elastic collision. Let us go back to the original equation with vectors.

The momentum conservation for the present case:

$$m\mathbf{v} = m\mathbf{u}_1 + m\mathbf{u}_2 \tag{3.27}$$

The mechanical energy conservation for the present case:

$$\frac{1}{2}m\mathbf{v}_1^2 = \frac{1}{2}m\mathbf{u}_1^2 + \frac{1}{2}m\mathbf{u}_2^2 \tag{3.28}$$

After some obvious mathematical simplifications:

$$\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 \quad \mathbf{v}_1^2 = \mathbf{u}_1^2 + \mathbf{u}_2^2 \tag{3.29}$$

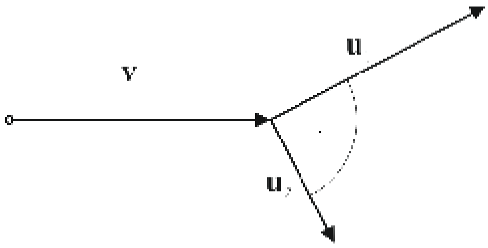


Figure 21: Billiard ball collision

First equation means that the vectors create a closed triangle. The second equation shows that the created triangle is a right angle triangle, since the Pythagoras theorem is true only then. As a summary, one can say that the balls travel 90 degree angle relative to each after collision in billiard game.

3.1.1.3 Ballistic pendulum

(Ballisztikus inga)

This is a pendulum with some heavy sand bag on the end of some meter long rope. The rope is hung on a high fix point, letting the pendulum swing. A simple indicator mechanism shows the highest angular excursion.

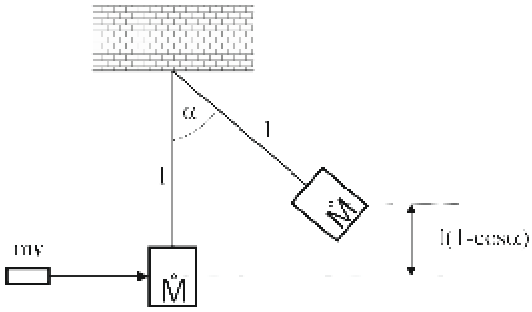


Figure 22: Ballistic pendulum

The pendulum is left to get quiet and hang vertically. Then the gun is fired, the bullet penetrates into the sandbag and get stuck in it. The pendulum starts to swing. The first highest angular excursion is detected. From the above information, the speed of the bullet can be found.

The whole process consists of two steps. In step 1 the bullet collides with the sandbag. Up to this point, conservation momentum is valid but the mechanical energy is not conserving quantity, due to the inelastic collision. After collision in step 2, there are no more dissipative effects, so conservation of mechanical energy is true. The two relevant equations are as follows:

$$mv = (m+M)u \tag{3.30}$$

$$\frac{1}{2}(m+M)u^2 = (m+M)gl(1-\cos\phi) \tag{3.31}$$

Here m and M are the mass of the bullet and the sandbag respectively.
The v and u are the speed of the bullet and the speed of the sandbag respectively.
The l is the length of the rope and g is the gravity acceleration.

By eliminating u from the equations one can readily express the incoming speed of the bullet:

$$\frac{m}{m+M}v = u \tag{3.32}$$

$$\frac{1}{2}(\frac{m}{m+M}v)^2 = gl(1-\cos\phi) \tag{3.33}$$

$$v = \frac{m+M}{m}\sqrt{2gl(1-\cos\phi)} \tag{3.34}$$

This is an excellent example how careful one must be. If wrongly the whole process is assumed to be conservative, the resulting bullet speed will be some ten meters per second which is roughly hundred times smaller than the real result.

3.1.2 Missile motion

(Rakéta mozgás)

Jet propulsion is the fundamental basis of the missile motion. This is based on the conservation of momentum. If one tries to hold the garden hose when sprinkling the garden, one will experience a recoil type force, which is pushing back. This force is called "thrust", and this drives the missiles, aircrafts and jet-skis.

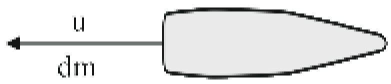


Figure 23: Missile motion

The missile ejects mass in continuous flow with the ejection speed (u) relative to the missile. The rate with which the mass is ejected is denoted (μ) and measured in kg/s. The infinitesimal ejected momentum will provide the impulse to the missile:

$$udm = Fdt \tag{3.35}$$

$$u\frac{dm}{dt} = F \tag{3.36}$$

$$u\mu = F \tag{3.37}$$

So the product of the speed and ejection rate determines the thrust (F). During the missile motion the thrust is a constant force. As the missile progresses the overall mass is continuously reduced by burning the fuel. The equation of motion is as follows:

$$F = m(t)a \tag{3.38}$$

Here $m(t)$ is the reducing mass and m_0 is the initial mass:

$$m(t) = (m_0 - \mu t) \tag{3.39}$$

$$u\mu = (m_0 - \mu t)a \tag{3.40}$$

The acceleration can be expressed:

$$\frac{u\mu}{m_0 - \mu t} = a(t) \tag{3.41}$$

In order to find out the velocity time function, the above formula needs to be integrated:

$$v(t) = \int_0^t a(t')dt' = \int_0^t \frac{u\mu}{m_0 - \mu t'}dt' \tag{3.42}$$

After the integration the velocity function is revealed:

$$v(t) = u \ln \frac{m_0}{m_0 - \mu t} \tag{3.43}$$

The final formula shows that approaching the m_0/μ time the speed grows to the infinity. This value can not be reached since there must be a payload on the missile.

3.2 Angular momentum in system of particles

(Tömegpont rendszer impulzus nyomatéka)

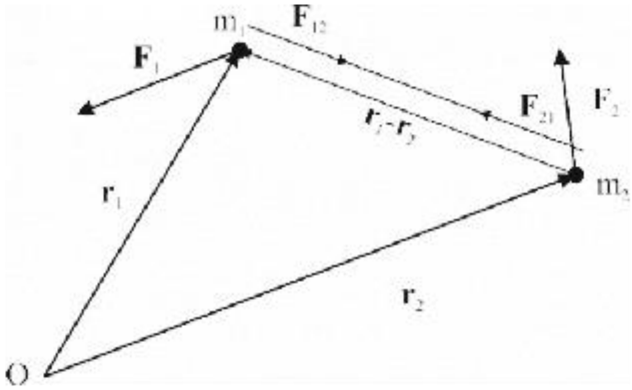


Figure 24: Angular momentum in system of particles

Consider the total angular momentum of a system of particles.

$$\mathbf{L}_1 = \mathbf{r}_2 \times m_1\mathbf{v}_1 \quad \mathbf{L}_2 = \mathbf{r}_2 \times m_1\mathbf{v}_2 \tag{3.44}$$

This is the sum of the angular momentums of the individual particles:

$$\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}_{tot} \tag{3.45}$$

Check out the time derivative oft this equation:

$$\frac{d\mathbf{L}_1}{dt} + \frac{d\mathbf{L}_2}{dt} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.46}$$

In Chapter 2 it has been shown that the derivative of angular momentum is the torque affecting the particle. Accordingly the above equation is transformed:

$$\mathbf{M}_1 + \mathbf{M}_2 = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.47}$$

Based on the definition of torque following formulas are true:

$$\mathbf{M}_1 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_1 \times \mathbf{F}_{12} \quad \mathbf{M}_2 = \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_2 \times \mathbf{F}_{21} \tag{3.48}$$

Let us substitute them to the equation above:

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_1 \times \mathbf{F}_{12} + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_2 \times \mathbf{F}_{21} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.49}$$

Here we have to use that $F_{12} = -F_{21}$ which is the consequence of action reaction principle.

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_1 \times \mathbf{F}_{12} + \mathbf{r}_2 \times \mathbf{F}_2 - \mathbf{r}_2 \times \mathbf{F}_{12} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.50}$$

Now regroup the left hand side:

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.51}$$

If now one looks at the figure above, the fact is readily apparent that the r_1-r_2 vector and F_{12} force vector are collinear vectors, thus their cross product is zero. Therefore the torques of the internal forces cancels out.

$$\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.52}$$

The left hand side terms are all the torques of the external forces. So all together the generalized statement is as follows:

$$\sum_i \mathbf{M}_{iext} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.53}$$

In words:

In system of particles, the time derivative of the total angular momentum is the sum of the external torques. This statement is called the theorem of angular momentum. Therefore the internal torques are ineffective in terms of total angular momentum.

If on the other hand the total external torque is zero, then the total angular momentum is constant. This is the conservation of angular momentum.

$$\mathbf{L}_{tot} = Const \tag{3.54}$$

In summary: In a system of particles where the total external torque is zero, the total angular momentum is constant, or in other words it is a conserving quantity.

This law can also be used in coordinate components. So if the system of particles is mounted on a bearing, and external torque parallel with the axis of the bearing does not affect the system, then that component of the angular momentum will be constant which is parallel with the axis of the bearing. In terms of other directions no any law applies.

3.2.1 The skew rotator

(Ferdeszögű forgás)

Consider the figure below. Two equal masses are placed on the ends of a weightless rod. The center of mass is mounted on a vertical axis, which is rotating freely in two bearings. The angle of the fixture is intentionally not ninety degrees, but a skew acute angle. The system rotates with a uniform angular velocity. The job is to find out the deviational torque which emerges, due to the rotation of asymmetric structure.

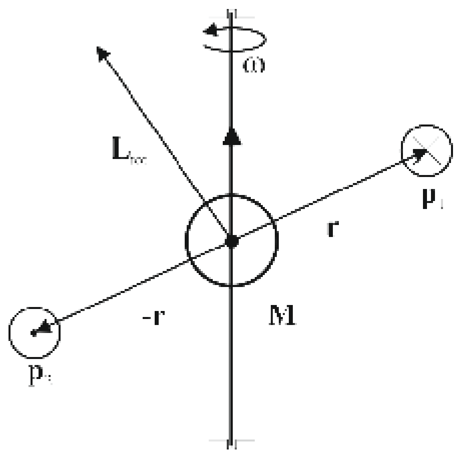


Figure 25: The skew rotator

The origin of the coordinate system is the center of mass. The coordinate system is not rotating together with the mechanical structure and it is considered inertial system. Gravity cancels out from the discussion, since the center of mass is supported by the axis, and the gravity does not affect torque to the system. The mechanical setup is in the plane of the figure. The two position vectors of the particles are (r) and (-r).

Momentums of the particles are p_1 and p_2. They are normal to the paper sheet.

$$\mathbf{p}_1 = m(\omega \times \mathbf{r}) \quad \mathbf{p}_2 = -m(\omega \times \mathbf{r}) \tag{3.55}$$

The corresponding L_1 and L_2 angular momentums are equal, because of the twice negative multiplication:

$$\mathbf{L}_1 = \mathbf{r} \times \mathbf{p}_1 = \mathbf{r} \times m(\omega \times \mathbf{r}) \quad \mathbf{L}_2 = -\mathbf{r} \times \mathbf{p}_2 = -\mathbf{r} \times m(\omega \times (-\mathbf{r})) \tag{3.56}$$

So the total angular momentum is the sum of these two:

$$\mathbf{L}_{tot} = 2\mathbf{r} \times m(\omega \times \mathbf{r}) \tag{3.57}$$

Now we use the theorem of angular momentum:

$$\sum_i \mathbf{M}_{iext} = \frac{d\mathbf{L}_{tot}}{dt} \tag{3.58}$$

$$\sum_i \mathbf{M}_{iext} = \frac{d}{dt}\left(2\mathbf{r} \times m(\omega \times \mathbf{r})\right) = 2\frac{d\mathbf{r}}{dt} \times m(\omega \times \mathbf{r}) + 2\mathbf{r} \times m(\omega \times \frac{d\mathbf{r}}{dt}) \tag{3.59}$$

Now take it into consideration that the derivative of the position is the velocity which can be expressed by means of angular velocity vector:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \omega \times \mathbf{r} \tag{3.60}$$

After substitution:

$$\sum_i \mathbf{M}_{iext} = 2(\omega \times \mathbf{r}) \times m(\omega \times \mathbf{r}) + 2\mathbf{r} \times m(\omega \times (\omega \times \mathbf{r})) \tag{3.61}$$

The first term on the right hand side is zero, because this is a cross product of collinear vectors. The final result comes up immediately.

$$\sum_i \mathbf{M}_{iext} = 2\mathbf{r} \times m(\omega \times (\omega \times \mathbf{r})) \tag{3.62}$$

After following the numerous cross products in terms of direction one can conclude, that the torque is pointing out of the paper sheet. The direction of torque is rotating together with the mechanical structure and always perpendicular to its plane. This is deviational torque and this emerges because the angular momentum vector is constantly changing, not in absolute value but in direction. This effect is very detrimental to the bearings due to the load that it generates. There are cases however when such effect does not show up. When the angular momentum vector is parallel with angular velocity vector no deviational torque will emerge. These are called principal axes. In general there are three perpendicular directions of principal axes.

A new interesting aspect:

Imagine that this experiment is carried out on a spacecraft orbiting the Earth. Suddenly the bearing and the mechanical axis disappear. How will the mass-rod-mass structure move after this?

Since no external torque affects the system, the angular momentum will be constant. But now the angular velocity vector starts to go around on the surface of a virtual cone. The symmetry axis of such virtual cone is just the angular momentum vector. This kind of motion is called precession.

3.2.2 The pirouette dancer (The symmetrical rotator)

(A piruett táncos)

In conjunction with the previous section this section could be called as "symmetrical rotator". The setup fundamentally similar, the major difference is that the mass-rod-mass system is positioned perpendicularly to the rotation axis.

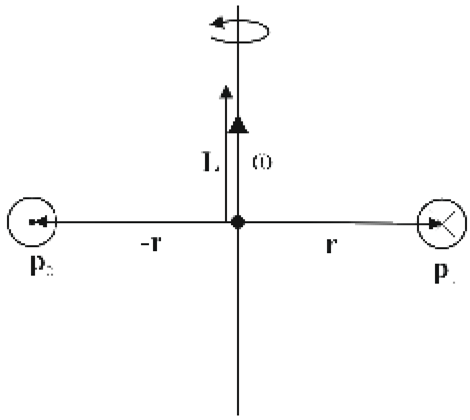


Figure 26: Symmetrical rotator